# Approximation Algorithms for the Weighted Independent Set Problem

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**Abstract.** In unweighted case, approximation ratio for the independent set problem has been analyzed in terms of the graph parameters, such as the number of vertices, maximum degree, and average degree. In weighted case, no corresponding results are given for average degree. It is not appropriate that we analyze weighted independent set algorithms in terms of average degree, since inserting the vertices with small weight decreases average degree arbitrarily without significantly changing approximation ratio. In this paper, we introduce the "weighted" average degree and "weighted" inductiveness, and analyze algorithms for the weighted independent set problem in terms of these parameters.

## 1 Introduction

An independent set in a graph is a set of vertices in which no two vertices are adjacent. The (weighted) independent set problem is that of finding a maximum (weight) independent set. There have been proposed and analyzed numerous approximation algorithms for this problem. In unweighted case, an algorithm with approximation ratio $\Delta/6 + O(1)$ was proposed by Halldórsson and Radhakrishnan [6] for the graphs with the maximum degree $\Delta$. Vishwanathan proposed the SDP-based algorithm whose approximation ratio is $O(\Delta \log\log \Delta / \log \Delta)$ [3]. For the graphs with the average degree $\overline{d}$, Hochbaum [7] proved that a version of Greedy algorithm has approximation ratio $(\overline{d}+1)/2$. Halldórsson and Radhakrishnan [5] improved this approximation ratio to $(2\overline{d}+3)/5$. Moreover, an algorithm with approximation ratio $O(\overline{d} \log\log \overline{d} / \log \overline{d})$ was proposed by Halldórsson [2]. In weighted case, Halldórsson and Lau [4] gave an algorithm with approximation ratio $(\Delta+2)/3$. For the $\delta$-inductive graphs approximation ratio $(\delta+1)/2$ is known due to Hochbaum [7], and Halldórsson [2] proposed an algorithm with approximation ratio $O(\delta \log\log \delta / \log \delta)$. Note that $\delta \leq \Delta$ for any graph.

In this paper, we extend the approximation algorithms of [2, 7] to the weighted case. Since inserting the vertices with small weight decreases $\overline{d}$ arbitrarily without significantly changing approximation ratio, we introduce the *weighted average degree* $\overline{d}_w$ and analyze the approximation ratio. For weighted graphs, there exist approximation algorithms whose approximation ratio is analyzed in terms of inductiveness. We extend inductiveness to weighted version and introduce the *weighted inductiveness* $\delta_w$.

The rest of this paper is organized as follows. In Section 2 we define the weighted average degree and the weighted inductiveness. We also show the relationship between various degrees. In Section 3 we propose a greedy algorithm whose lower bound is $\max(W/(\overline{d}_w+1), W/(\delta_w+1))$, where $W$ is the total weight. We also prove that this algorithm has approximation ratio $\delta_w$. In Section 4 we prove that the approximation ratio of $\min((\overline{d}_w+1)/2, (\delta_w+1)/2)$ can be achieved. Finally we will prove that the approximation ratios of $O(\overline{d}_w \log\log \overline{d}_w / \log \overline{d}_w)$ and $O(\delta_w \log\log \delta_w / \log \delta_w)$ can be achieved in Section 5. We will assume that the input graphs have no isolated vertices.

## 2 Preliminaries

### 2.1 Definitions

Let $G$ be an undirected graph where each vertex $v$ has positive weight $w_v$. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively, as usual. Let $W(G)$ be the sum of the weights of all vertices. $n(G)$ is the number of vertices in $G$. Let $\Delta(G)$ and $\overline{d}(G)$ denote the maximum and the average degree of $G$, respectively. $d(v, G)$ is the degree of vertex $v$ in $G$. The inductiveness $\delta(G)$ of a graph $G$ is given by

$$\delta(G) = \max_{H \subseteq G} \min_{v \in V(H)} d(v, H), \tag{1}$$

where $H \subseteq G$ denotes that $H$ is a subgraph of $G$. Let $\pi$ be an ordering of vertices in $V$, that is, a one to one map $V \to \{1, 2, \ldots, n\}(n = |V|)$. We define the right degree of a vertex $v$ in $G$ with respect to $\pi$ as follows:

$$d^{\pi}(v, G) = |\{u \in V | (u, v) \in E, \pi(u) > \pi(v)\}|. \tag{2}$$

The right degree of a vertex $v$ is the number of adjacent vertices to the right when we arrange vertices from left to right according to $\pi$. If there exists $\pi$ such that $m \geq \max_v d^{\pi}(v, G)$, we call $G$ an $m$-inductive graph.

For a vertex set $X$, let $w(X)$ denote the sum of the weights of the vertices in $X$. Let $N_G(v)$ denote the set of vertices adjacent to vertex $v$ in $G$. For a vertex $v$, we define the weighted degree $d_w(v, G)$ in $G$ as follows:

$$d_w(v, G) = \frac{w(N_G(v))}{w_v}. \tag{3}$$

$\Delta_w(G) = \max_v d_w(v, G)$ is the maximum weighted degree of $G$. We will omit $G$ if it is clear from the context. We define the weighted average degree $\overline{d}_w(G)$ of graph $G$ as follows:

$$\overline{d}_w(G) = \frac{\sum_{v \in V} w_v d(v)}{W}. \tag{4}$$

In fact, we can represent the weighted average degree in the following form:

$$\overline{d}_w(G) = \frac{\sum_{v \in V} w(N(v))}{W} \quad (5)$$

$$= \frac{\sum_{v \in V} w_v d_w(v)}{W}. \quad (6)$$

The weighted inductiveness $\delta_w(G)$ of a graph $G$ is given by

$$\delta_w(G) = \max_{H \subseteq G} \min_{v \in V(H)} d_w(v, H). \quad (7)$$

We define the right weighted degree of a vertex $v$ for an ordering $\pi$ in $G$ as follows:

$$d_w^\pi(v, G) = \frac{w(\{u \in V | (u, v) \in E, \pi(u) > \pi(v)\})}{w_v}.$$

If there exists $\pi$ such that $m \geq \max_v d_w^\pi(v, G)$, we call $G$ a weighted $m$-inductive graph.

We denote $\alpha_w(G)$ as the weight of the optimal solution of the weighted independent set problem on $G$. For an algorithm $A$, $A(G)$ denotes the weight of the independent set obtained by $A$ on $G$. Then the approximation ratio of $A$ is defined by

$$\sup_G \frac{\alpha_w(G)}{A(G)}.$$

We will consider unweighted graphs as weighted ones where each vertex has unit weight. $\alpha(G)$ denotes the size of a maximum independent set on $G$.

### 2.2 Weighted inductiveness

Let $\pi$ be an ordering of the vertices of $G$ and $v_i$ be a vertex with $\pi(v_i) = i$. We define $V_i^\pi = \{v_j | j \geq i\}$. Let $G_i^\pi$ be the induced subgraph of $G$ by $V_i^\pi$. Smallest-first ordering $\pi$ is an ordering such that the weighted degree of $v_i$ is minimum in $G_i^\pi$ for all $i$ $(1 \leq i \leq n)$. We can find a smallest-first ordering in polynomial time. We can prove the following theorem in the same manner as in the case of unweighted inductiveness [8].

**Theorem 1.** *For any ordering $\pi$, the inequality*

$$\delta_w(G) \leq \max_v d_w^\pi(v, G)$$

*holds. Moreover, if $\pi$ is a smallest-first ordering, then the equality*

$$\delta_w(G) = \max_v d_w^\pi(v, G)$$

*holds.*

**Corollary 1.** *A smallest-first ordering $\pi$ minimizes* $\max_v d_w^\pi(v, G)$.

### 2.3 Relationship between weighted and unweighted degrees

**Theorem 2.** *The following relationships hold for all graphs $G$:*

$$\delta \leq \Delta_w \tag{8}$$

$$\delta_w \leq \Delta \tag{9}$$

$$\overline{d} \leq \Delta_w \tag{10}$$

$$\overline{d}_w \leq \Delta. \tag{11}$$

*Proof.* We can prove inequalities (8) and (9) straightforward by considering the non-decreasing order and the non-increasing order of weight, respectively. (11) follows immediately from the definition of measures. Finally, we prove inequality (10). We can get the following inequalities:

$$\sum_{v \in V} d_w(v) = \sum_{v \in V} \sum_{u:(u,v) \in E} \frac{w_u}{w_v} = \sum_{(u,v) \in E} \left[ \frac{w_u}{w_v} + \frac{w_v}{w_u} \right] \geq 2|E| = n\overline{d}.$$

Thus,

$$\Delta_w = \max_{v \in V} d_w(v) \geq \frac{1}{n} \sum_{v \in V} d_w(v) \geq \overline{d}.$$

Hence, this theorem holds. □

## 3 Greedy algorithm

### 3.1 Previous results

For unweighted graphs, the greedy algorithm can be written as follows. We select a minimum degree vertex as a vertex in the independent set $I$, and delete this vertex and all of its neighbors from the graph. We repeat this process for the remaining subgraph until the subgraph becomes empty. This algorithm attains the Turán bound [5, 7];

$$|I| \geq \frac{n}{\overline{d} + 1}. \tag{12}$$

For weighted graphs, there exists an algorithm which attains the following lower bound [2, 8]

$$w(I) \geq \frac{W}{\delta + 1}. \tag{13}$$

### 3.2 Algorithm for the weighted graphs

Our greedy algorithm for the weighted graphs is almost the same as the unweighted greedy algorithm. The difference is that, instead of selecting a minimum degree vertex, our algorithm selects a minimum weighted degree vertex. We call this algorithm WG.

### 3.3 Lower bound

We use the following proposition.

**Proposition 1.** *Assume that $a_i > 0$, $b_i > 0$ for all $1 \le i \le n$. Then the inequality*

$$\sum_i \frac{b_i^2}{a_i} \ge \frac{\left(\sum_i b_i\right)^2}{\sum_i a_i}$$

*holds.*

*Proof.* The inequality is equivalent to

$$\sum_i a_i \sum_i \frac{b_i^2}{a_i} \ge \left(\sum_i b_i\right)^2.$$

This inequality comes from the Cauchy-Schwarz inequality $\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right) \ge \left(\sum_i x_i y_i\right)^2$, by assigning $x_i = \sqrt{a_i}$ and $y_i = b_i/\sqrt{a_i}$. □

Let $I$ be the independent set obtained by $\mathsf{WG}$. Let $v_i$ be the $i$-th vertex selected into the independent set $I$. Let $G_i$ be the subgraph induced by the remaining vertices at the beginning of the $i$-th iteration.

**Theorem 3.** $\mathsf{WG}$ *produces the independent set satisfying the inequality*

$$\mathsf{WG}(G) \ge \frac{W}{\overline{d}_w + 1}.$$

*Proof.* We first argue the lower bound of $\overline{d}_w W$ as follows:

$$\begin{aligned}\overline{d}_w W &= \sum_{v \in V(G)} w_v d_w(v, G) \ge \sum_i \sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v d_w(v, G_i) \\ &\ge \sum_i \sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v d_w(v_i, G_i) = \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right) d_w(v_i, G_i).\end{aligned}$$

Adding $W = \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right)$, we can deduce the inequality

$$\left(\overline{d}_w + 1\right) W \ge \sum_i \frac{\left(w(N_{G_i}(v_i)) + w_{v_i}\right)^2}{w_{v_i}}.$$

Finally we apply Proposition 1 with $a_i = w_{v_i}$, $b_i = w(N_{G_i}(v_i)) + w_{v_i}$. The inequality

$$\left(\overline{d}_w + 1\right) W \ge \frac{W^2}{\mathsf{WG}(G)}$$

holds, which implies the theorem. □

Note that $\mathsf{WG}$ can find an independent set with the following lower bound [9]:

$$\mathsf{WG}(G) \geq \sum_{v \in V} \frac{w_v^2}{w(N(v)) + w_v}.$$

This lower bound also leads to Theorem 3.

**Theorem 4.** $\mathsf{WG}$ *produces the independent set satisfying the inequality*

$$\mathsf{WG}(G) \geq \frac{W}{\delta_w + 1}.$$

*Proof.* Because $\delta_w \geq d_w(v_i, G_i)$ for all $i$ and $W = \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right)$, the inequality

$$W\delta_w \geq \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right) d_w(v_i, G_i)$$

holds. With this inequality, we can prove this theorem in the same way as Theorem 3. □

**Proposition 2.** *The lower bounds of Theorems 3 and 4 are tight.*

*Proof.* We illustrate the tight example for both theorems. Let $G$ be a star graph with $n$ vertices. We assign weight $w$ to the central vertex and $w/\sqrt{n-1}$ to the other vertices. In this graph, $\overline{d}_w = \delta_w = \sqrt{n-1}$, $W = (\sqrt{n-1}+1)w$. $\mathsf{WG}$ may output the singleton with the central vertex. In this case, $\mathsf{WG}(G) = w$ and thus the inequalities in Theorems 3 and 4 hold with equality. □

### 3.4 Approximation ratio

**Theorem 5.** $\mathsf{WG}$ *attains approximation ratio* $\delta_w$.

*Proof.* Let $V_i = N_{G_i}(v_i) \cup \{v_i\}$, and $H_i$ be the induced subgraph by $V_i$ of $G$. We will prove the inequality $\alpha_w(H_i) \leq w_{v_i}\delta_w$. $v_i$ is adjacent to all other vertices in $H_i$, so $\alpha_w(H_i) \leq \max(w_{v_i}, w(N_{H_i}(v_i)))$. By the property of $\mathsf{WG}$ and the definition of the weighted inductiveness, the inequality $d_w(v_i, H_i) \leq \delta_w$ holds. By the definition of weighted degree, the inequality $\max(w_{v_i}, w(N_{H_i}(v_i))) \leq w_{v_i}\delta_w$ holds. Thus, $\alpha_w(H_i) \leq w_{v_i}\delta_w$ is proved. The inequalities

$$\alpha_w(G) \leq \sum_i \alpha_w(H_i) \leq \sum_i w_{v_i}\delta_w = \mathsf{WG}(G)\delta_w$$

are immediate. □

**Proposition 3.** *The approximation ratio* $\delta_w$ *of* $\mathsf{WG}$ *is tight.*

*Proof.* The graph in Proposition 2 is the tight example. □

# 4 Linear programming algorithm

## 4.1 Unweighted results

We will consider the combination of linear programming and the greedy algorithm. With the lower bound (12), Hochbaum [7] proved that this combination achieves the approximation ratio $(\overline{d}+1)/2$. In this section we extend Hochbaum's analysis to the weighted case and prove that the proposed algorithm has the approximation ratios $(\overline{d}_w+1)/2$ and $(\delta_w+1)/2$.

## 4.2 LP relaxation for the weighted independent set problem

The weighted independent set problem can be formulated in the integer programming as follows:

$$\begin{array}{rll} \text{maximize} & \sum_{i \in V} w_i x_i, & \\ \text{subject to} & x_i + x_j \leq 1 & \text{for all } (i,j) \in E, \\ & x_i \in \{0,1\} & \text{for all } i \in V. \end{array} \tag{14}$$

Relaxing the integral constraint, we can deduce the following linear programming:

$$\begin{array}{rll} \text{maximize} & \sum_{i \in V} w_i x_i, & \\ \text{subject to} & x_i + x_j \leq 1 & \text{for all } (i,j) \in E, \\ & 0 \leq x_i \leq 1 & \text{for all } i \in V. \end{array} \tag{15}$$

We can obtain the optimal solution to this LP each of whose elements is 0, 1/2, or 1 [10]. We classify the vertices into three sets according to the value of $x_i$, that is, $S_1 = \{i \in V | x_i = 1\}$, $S_{1/2} = \{i \in V | x_i = 1/2\}$, $S_0 = \{i \in V | x_i = 0\}$. Note that $S_1$ is an independent set of $G$ and no vertex in $S_{1/2}$ has a neighbor in $S_1$. We also note that $S_{1/2}$ induces the subgraph with no isolated vertices.

## 4.3 Algorithm

We first solve the LP relaxation to divide the vertex set $V$ into three subsets $S_1$, $S_{1/2}$, and $S_0$ as above. We then apply WG to the subgraph $H$ induced by $S_{1/2}$ to obtain an independent set $I_H$ of $H$. Finally, we output the independent set $I = S_1 \cup I_H$. We call this algorithm WGL.

## 4.4 Approximation ratio

From Theorem 3, we can prove the following theorem in the same manner as the Hochbaum's proof [7] of the approximation ratio $(\overline{d}+1)/2$ for unweighted graphs.

**Theorem 6.** *Approximation ratio of* WGL *is* $(\overline{d}_w+1)/2$.

We prove the approximation ratio in terms of the weighted inductiveness.

**Theorem 7.** *Approximation ratio of* $\mathsf{WGL}$ *is* $(\delta_w + 1)/2$.

*Proof.* From Theorem 4,

$$\frac{\alpha_w(G)}{\mathsf{WGL}(G)} \le \frac{w(S_1) + \frac{1}{2}w(S_{\frac{1}{2}})}{w(S_1) + \frac{w(S_{\frac{1}{2}})}{\delta_w(H)+1}} \le \frac{\delta_w(H)+1}{2} \le \frac{\delta_w+1}{2}. \qquad \square$$

**Proposition 4.** *The approximation ratio of Theorem 6 and 7 is tight.*

*Proof.* We consider the split graph $G = (V, E)$, where $V = \{u_1, u_2, \ldots, u_t, v_1, v_2, \ldots, v_{2t-1}\}$ and $E = \{(u_i, v_j) | 1 \le i \le t, 1 \le j \le 2t-1\} \cup \{(u_i, u_j) | 1 \le i < j \le t\}$. The induced subgraph by $\{u_i\}$ is a clique and the set $\{v_i\}$ is an independent set. We give each vertex $u_i$ weight $w/t+\epsilon$, each vertex $v_i$ weight $w/(2t-1)$, where $\epsilon$ is a small positive constant. In the optimal solution for LP (15), each value of $x_i$ is $1/2$. Thus, $S_{1/2} = V(G)$. In this graph, $\mathsf{WGL}(G) = w/t + \epsilon$ and $\alpha_w = w$. So, the following equations hold:

$$\overline{d}_w = 2t - 1 + \frac{3t^2 - 2t}{2w}\epsilon, \qquad \frac{\alpha_w(G)}{\mathsf{WGL}(G)} = \frac{\overline{d}_w + 1}{2} - \left(\frac{t^2}{w + \epsilon t} - \frac{3t^2 - 2t}{4w}\right)\epsilon,$$

$$\delta_w = 2t - 1 - \frac{t^2}{w + \epsilon t}\epsilon, \qquad \frac{\alpha_w(G)}{\mathsf{WGL}(G)} = \frac{\delta_w + 1}{2} - \frac{t^2}{2(w + \epsilon t)}\epsilon.$$

Hence, Theorems 6 and 7 are tight. $\square$

## 5 Semi-definite programming

### 5.1 Previous result

The following theorem was proved in [2]:

**Theorem 8.** *For any fixed real* $k$ *such that* $\vartheta_w(G) \ge 2W/k$, *we can construct an independent set in* $G$ *whose weight is* $\Omega(W/(k\delta^{1-1/(2k)}))$.

The function $\vartheta_w(G)$, defined in [1], is the weighted version of Lovász's $\vartheta$-function. This function can be computed using a semi-definite programming (SDP) in polynomial time, and has the property $\alpha_w(G) \le \vartheta_w(G)$.

For the unweighted graphs, the combination of this theorem and the greedy algorithm yields the approximation ratio $O(\overline{d} \log \log \overline{d} / \log \overline{d})$.

### 5.2 Approximation ratio for the weighted graphs

We will prove the following result for the weighted version of the algorithm with the approximation ratio $O(\overline{d} \log \log \overline{d} / \log \overline{d})$.

**Theorem 9.** *For any fixed real* $t$ *such that* $t \ge W(G)/\alpha_w(G)$, *we can approximate the weighted independent set problem within* $O(t^2 \overline{d}_w^{1-1/(8t)})$.

*Proof.* Assume that $t \geq W(G)/\alpha_w(G)$ is fixed. Let $K$ be the subgraph induced by the vertices whose degrees in $G$ are less than $2t\overline{d}_w$. Then we can estimate the value $\overline{d}_w W(G)$ as follows:

$$\overline{d}_w W(G) = \sum_{v \in V(G)} w_v d(v) \geq \sum_{v \in V(G) \setminus V(K)} w_v d(v) \geq 2t\overline{d}_w \sum_{v \in V(G) \setminus V(K)} w_v.$$

Thus, the inequality $\sum_{v \in V(G) \setminus V(K)} w_v \leq W(G)/(2t)$ holds. From this inequality, we can prove the theorem along with [2]. □

**Theorem 10.** *For any fixed real $t$ such that $t \geq W(G)/\alpha_w(G)$, we can approximate the weighted independent set problem within $O(t^2 \delta_w^{1-1/(8t)})$.*

*Proof.* Let $\pi$ be an ordering of vertices in $G$ with which the value of $\max_v d_w^\pi(v)$ is equal to $\delta_w$. Let $\pi'$ be the reverse ordering of $\pi$. Assume that $t \geq W(G)/\alpha_w(G)$ is fixed. Let $K$ be the subgraph induced by the vertices whose right degrees $d^{\pi'}(v, G)$ are less than $2t\delta_w$. Thus $K$ is a $2t\delta_w$-inductive graph. Then the following inequalities hold:

$$\begin{aligned} W\delta_w &\geq \sum_{v \in V(G)} w_v d_w^\pi(v) = \sum_{v \in V(G)} w_v d^{\pi'}(v) \\ &\geq \sum_{v \in V(G) \setminus V(K)} w_v d^{\pi'}(v) \geq 2t\delta_w \sum_{v \in V(G) \setminus V(K)} w_v. \end{aligned}$$

Thus, we can prove this theorem just like [2]. □

### 5.3 Algorithm

In this section we propose two algorithms: WGSA, whose approximation ratio is a function of $\overline{d}_w$, and WGSI, whose approximation ratio is a function of $\delta_w$.

WGSA is the following algorithm. We get an independent set by applying WG. Independently, we apply the algorithm given by Theorem 9 to obtain another independent set. We output the one with larger weight.

**Theorem 11.** WGSA *can achieve approximation ratio $O(\overline{d}_w \log\log \overline{d}_w / \log \overline{d}_w)$ for the weighted independent set problem.*

*Proof.* From Theorems 3 and 9, we can prove this theorem in the same manner as [2]. □

WGSI is the following algorithm. We get an independent set by applying WG. Independently, we apply the algorithm given by Theorem 10 to obtain another independent set. We output the one with larger weight.

**Theorem 12.** WGSI *can achieve approximation ratio $O(\delta_w \log\log \delta_w / \log \delta_w)$ for the weighted independent set problem.*

*Proof.* From Theorems 4 and 10, we can prove this theorem in the same way as [2]. □

## 6 Conclusion

In this paper, we defined the weighted average degree $\overline{d}_w$ and the weighted inductiveness $\delta_w$, and proved the lower bound of the weight of the independent set obtained by the weighted greedy algorithm. We also proved that this algorithm has approximation ratio $\delta_w$. Combining with LP, we obtained the approximation ratio $\min((\overline{d}_w+1)/2, (\delta_w+1)/2)$. Also combining with SDP, we proved that approximation ratio can attain $O(\overline{d}_w \log\log \overline{d}_w / \log \overline{d}_w)$ and $O(\delta_w \log\log \delta_w / \log \delta_w)$.


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## References


1. M. Grötschel, L. Lovász, A. Schrijver. *Geometric algorithms and combinatorial optimization, 2nd ed.* Springer-Verlag, 1993.
2. M.M. Halldórsson. Approximations of weighted independent set and hereditary subset problems. *Journal of Graphs Algorithms and Applications,* 4(1):1–16, 2000.
3. M.M. Halldórsson. Approximations of independent sets in graphs. In K. Jansen, J. Rolim, editors, *The First International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX),* 1–14, 1998.
4. M.M. Halldórsson and H.C. Lau. Low-degree graph partitioning via local search with applications to constraint satisfaction, max cut, and 3-coloring. *Journal of Graph Algorithms and Applications,* 1(3):1–13, 1997.
5. M.M. Halldórsson and J. Radhakrishnan. Greed is good: Approximating independent sets in sparse and bounded-degree graphs. *Algorithmica,* 18:145–163, 1997.
6. M.M. Halldórsson and J. Radhakrishnan. Improved approximations of independent sets in bounded-degree graphs via subgraph removal. *Nordic Journal of Computing,* 1(4):475–482, 1994.
7. D.S. Hochbaum. Efficient bounds for the stable set, vertex cover and set packing problems. *Discrete Applied Mathematics,* 6:243–254, 1983.
8. D.W. Matula and L.L. Beck. Smallest-last ordering and clustering and graph coloring algorithms. *Journal of the Association for Computing Machinery,* 30(2):417–427, 1983.
9. S. Sakai, M. Togasaki, K. Yamazaki. A note on greedy algorithms for maximum weighted independent set problem. *Discrete Applied Mathematics,* 126:313–322, 2003.
10. V.V. Vazirani. *Approximation algorithms.* Springer-Verlag, 2001.