On Powers of Chordal Graphs
And Their Colorings

Geir Agnarsson[*]   *Raymond Greenlaw*[†]
Magnús M. Halldórsson[*]

Abstract

The k-th *power* of a graph G is a graph on the same vertex set as G, where a pair of vertices is connected by an edge if they are of distance at most k in G. We study the structure of powers of chordal graphs and the complexity of coloring them. We start by giving new and constructive proofs of the known facts that any power of an interval graph is an interval graph, and that any odd power of a general chordal graph is again chordal. We then show that it is computationally hard to approximately color the even powers of n-vertex chordal graphs within an $n^{\frac{1}{2}-\epsilon}$ factor, for any $\epsilon > 0$. We present two exact and closed formulas for the chromatic polynomial for the k-th power of a tree on n vertices. Furthermore, we give an $O(kn)$ algorithm for evaluating the polynomial.

Keywords: Chordal graphs, chromatic number, chromatic polynomial, coloring, interval graphs, power of a graph, tree.

1   Introduction

In this paper we study the structure of powers of chordal graphs and some important subclasses. (Background material and most classes of graphs discussed in this paper are defined in §2.) Specifically, we simplify the proofs of several known theorems, make the proofs more constructive, and sometimes generalize the results. The intent is to penetrate the characteristic properties of chordal graphs and their powers by elementary methods, and show how even powers of chordal graphs are harder to work with than odd powers. In addition, we consider colorings of the special class of *chordal*

[*]Science Institute, University of Iceland, Reykjavík, Iceland. {geira,mmh}@hi.is

[†]Department of Computer Science, Armstrong Atlantic State University, 11935 Abercorn Street, Savannah, Georgia 31419-1997. greenlaw@pirates.armstrong.edu

trees formed by taking powers of trees, and give two exact formulas for their chromatic polynomial.

Chordal graphs and colorings of chordal graphs have been studied intensively. Many important subclasses of chordal graphs such as trees, interval graphs, and strongly chordal graphs have received special attention. We note that every strongly chordal graph and every split graph is chordal. Also, every tree and every interval graph is strongly chordal.

The fact that any power of a tree is chordal and proof thereof appears in [16] and [4]. However, Robert Jamison (Personal Communication, 2000) at Clemson University may have been the first to prove that property in the early eighties. Linear time algorithms are given in [16] for finding a tree square root of a given graph and a square root of a planar graph. In [4] a polynomial time algorithm for recognizing tree powers is given as well as a short proof that any power of a tree is strongly chordal. In [19] and [5], it is shown that any power of a strongly chordal graph is again chordal. In [18] it is shown that any power of an interval graph is again interval. A characterization of strongly chordal graphs in terms of totally balanced matrices is given in [8], where strongly chordal graphs were first introduced.

The subclass of chordal graphs consisting of strongly chordal graphs has been researched thoroughly. This is mainly because they yield a polynomial time solvability of the domatic set and domatic partition problems. The interested reader may pursue such results by following references [2, 9, 13, 17].

Some fundamental properties have been proved for arbitrary chordal graphs. It was shown that in [14] that if G is chordal then G^3 and G^5 are chordal, while G^2 is not necessarily chordal. There it was conjectured that any odd power of a chordal graph is again chordal. This conjecture was proved in [1], although Duchet had shown this earlier in a different setting [7]. In [15] a necessary and sufficient condition is given in order for all powers of a chordal graph to be chordal. Other subclasses of perfect graphs have been shown to be closed under taking powers, including the class of cocomparability graphs (which includes interval graphs) [6].

The remainder of this paper is organized as follows. In §2 we introduce our notation, prove some useful lemmas, and give an elementary and self-contained proof of the fact that any power of a tree is chordal.

In §3 we prove constructively that any power of an interval graph is again an interval graph. We also give a linear time algorithm to construct the power graph. The construction additionally applies to circular-arc graphs, which as a result are also closed under taking powers. We close the section by giving a direct and alternative proof of the fact that any odd power of a chordal graph is chordal.

In §4 we show that it is hard to approximately color even powers of chordal graphs within $O(n^{\frac{1}{2}-\epsilon})$, for any $\epsilon > 0$, unless NP-problems have

2

randomized polynomial time algorithms. This is essentially the strongest result possible, since greedy colorings of arbitrary even power graphs are $\sqrt{n}$-approximate. We study odd powers of general graphs, and also find them hard to color approximately within an $n^{1/2-\epsilon}$ factor, for any $\epsilon > 0$, however, we give a greedy $n^{2/3}$-approximation.

Lastly, in §5 we study powers of trees and colorings of those powers. We give two exact and closed formulas for the chromatic polynomial of the k-th power of a tree with n vertices. We complement these precise formulas with an $O(kn)$ algorithm to evaluate the chromatic polynomial at any point.

2   Preliminaries

For a set S and an element s, denote by $S \setminus \{s\}$ the set containing all elements of S excluding s. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $Z^+ = \mathbb{N} \setminus \{0\}$. For $a \leq b \in Z^+$, we denote the set $\{a, a+1, \ldots, b\}$ by $[a; b]$. If a equals 1, then we simply write $[b]$ instead of $[1; b]$.

All graphs in this paper are finite, simple, and undirected unless otherwise stated. By *coloring* we will always mean the usual vertex coloring of a given graph.

For a graph G, we denote the set of vertices of G by $V(G)$, and the set of edges of G by $E(G)$. We reserve the symbol n for $|V(G)|$, the number of vertices of G. For $v \in V(G)$, let $N(v)$ denote the *neighborhood* of v, that is, the set of all vertices adjacent to v but not including v itself. Likewise, let $N[v]$ denote the *closed neighborhood* of v, which is the set that consists of all the vertices adjacent to v, together with v itself. For a subgraph H of G, the graph in G *generated* or *induced* by H, denoted $G[H]$, is the subgraph which has vertex set $V(H)$, and edge set $\{\{u, v\} : u, v \in V(H) \text{ and } \{u, v\} \in E(G)\}$. If $U \subseteq V(G)$ then $G[U]$ is the subgraph induced by the subgraph having vertex set U and an no edges. For an edge $\{u, v\}$, the vertices u and v are called the *endvertices* of the edge. The *distance* between u and v in G is the number of edges in the shortest path from u to v. For two vertices u and v in a graph G, we denote the distance between u and v in G by $d_G(u, v)$. This is shortened to $d(u, v)$ when G is understood.

For a graph G and an edge e of G, let $G \setminus e$ denote the graph obtained by deleting e from G. On the other hand, let G/e denote the simple contraction of G by the edge e.

We call an edge a *chord* of a simple cycle of length four or more, if it is not in the cycle, but has both its endvertices on the cycle. A graph G is called *chordal* if every simple cycle of length four or more has a chord. In fact, there is an equivalent and more computational condition for a graph being chordal. A graph G is chordal if and only if there is an ordering $\{v_1, \ldots, v_n\}$ on $V(G)$, called a *simplicial elimination ordering*, such that

for each v_i, the set $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ induces a clique in G [20, Theorem 5.3.13, page 199].

An *odd chord* is an edge joining two vertices that are an odd distance apart in a cycle. A chordal graph is *strongly chordal* if every simple cycle of even length six or more has an odd chord. An equivalent condition is that there exist an ordering $\{v_1, \ldots, v_n\}$ on $V(G)$, called a *strong elimination ordering*, which is a simplicial elimination ordering such that if $k < j < i$, and $v_k, v_j \in N(v_i)$, then

$$N(v_k) \cap \{v_1, \ldots, v_{i-1}\} \subseteq N(v_j) \cap \{v_1, \ldots, v_{i-1}\} \quad [8, 13].$$

An *interval graph* is a graph whose vertex set can be represented by a collection of proper, closed intervals of the real numbers, and where two vertices are connected by an edge if and only if the corresponding intervals have a nonempty intersection. It is easy to see that interval graphs are a subclass of chordal graphs. A graph G is called a *split graph*, if $V(G)$ can be partitioned into a disjoint union $X \cup Y$ where the induced subgraph $G[X]$ has no edges, and $G[Y]$ is complete.

For $k \in Z^+$ and a graph G, the *k-th power* of G, denoted by G^k, is the graph formed from $V(G)$, where all pairs of vertices having distance of length k or less in G are connected by an edge. Note that the original edges in G are retained.

Let T be an unrooted tree and let l be the length of the longest path in T (that is, l is the *diameter* of T). A vertex of degree one in T is called a *leaf*. If l is even, then the *centroid of T* is the unique vertex, which is of distance at most $l/2$ from all the leaves of T. If l is odd, then the *centroid of T* is the unique adjacent pair of vertices such that each leaf is of distance at most $(l-1)/2$ from one of these two vertices. In a tree T, rooted at vertex r, an *ancestor* of v is any vertex on the (unique) path from v to r. Note, v is an ancestor of itself.

For a tree T and $k \in Z^+$, Lin and Skiena show that the graph T^k is chordal [16, §5]. Their proof depends heavily on the characterization of chordal graphs as precisely the *intersection graphs* (graphs formed from a collection of sets, where the sets represent the vertices and two vertices are adjacent if and only if their corresponding sets intersect) of subtrees in trees [11]. Corneil and Kearney later proved that any power of a tree is actually strongly chordal [4].

We conclude this section with a definition, followed by two lemmas, that allow us to present an elementary and new proof of the Lin and Skiena result.

Definition 2.1 *For a graph G and $k \in \mathbb{N}$, we define a k-ball as a subset $B \subseteq V(G)$, such that for any vertices $u, v \in B$, we have $d_G(u, v) \leq k$.*

4

Lemma 2.2 *Let T be a tree rooted at vertex r. For every vertex $u \in V(T)$, the set of vertices $v \in V(T)$ satisfying*

$$
\begin{aligned}
d_T(u, v) &\leq k \ and \\
d_T(r, v) &\leq d_T(r, u),
\end{aligned}
\tag{1}
$$

is a k-ball. That is, the vertices which are closer to the root than u, and are of distance k or less from u, form a k-ball in T.

Proof. We want to show that any two vertices v and v' satisfying the conditions stated in the lemma are of distance k or less from each other. Let w (w') be the least common ancestor of u and v (respectively, u and v'). Since both w and w' are ancestors of u, either w is an ancestor of w' or vice-versa. Without loss of generality, suppose w' is an ancestor of w. Since v satisfies (1), we have $d_T(u, w) \geq d_T(v, w)$. Note that in a tree a simple path between two vertices is the shortest path between them. With this fact in mind, we have

$$
d_T(u, v') = d_T(u, w) + d_T(w, w') + d_T(w', v')
$$

Therefore,

$$
\begin{aligned}
d_T(v, v') &= d_T(v, w) + d_T(w, w') + d_T(w', v') \\
&\leq d_T(u, w) + d_T(w, w') + d_T(w', v') \\
&\leq d_T(u, v') \\
&\leq k.
\end{aligned}
$$

This proves the lemma. $\qquad\square$

Lemma 2.3 *If T is a tree and $k \in Z^+$, then T^k is chordal.*

Proof. Consider m distinct vertices $u_1, \ldots, u_m$ of T with the property that for each $i \in [m]$, the unique path $p(u_i, u_{i+1})$ from u_i to u_{i+1} has length k or less, where $u_{m+1} = u_1$. We show there is a $j \in [m]$ such that $d_T(u_{j-1}, u_{j+1}) \leq k$. This will imply an edge between u_{j-1} and u_{j+1} in T^k, and hence the chordality of T^k. Consider the subtree of T defined by

$$
H = p(u_1, u_2) \cup \cdots \cup p(u_{m-1}, u_m) \cup p(u_m, u_1).
$$

All the leaves of H are contained in the set $\{u_1, \ldots, u_m\}$. Let $r \in V(H)$ be a centroid of H. Root H at r and let j be such that $d_H(u_j, r) \geq d_H(u_i, r)$ for every $i \in [m]$. By Lemma 2.2 we see that $B_j = \{u : d_H(u, u_j) \leq k\}$ is a k-ball in H. Since u_{j-1} and u_{j+1} are contained in B_j, and $d_T = d_H$ when restricted to vertices in H, we have that $d_T(u_{j-1}, u_{j+1}) \leq k$. $\qquad\square$

The *clique size* of a graph is the number of vertices of the largest complete subgraph of the graph. Let $\omega(G)$ denote the clique size of G. By a theorem of Berge from 1960 [20, page 201], all chordal graphs G are *perfect*, meaning that $\chi(H) = \omega(H)$ for every induced subgraph H of G. In particular, $\chi(G) = \omega(G)$. The largest clique in T^k is induced by the largest k-ball in T. If $b_k(T)$ is the cardinality of a largest k-ball, then we get the following corollary.

Corollary 2.4 *For a tree T and $k \in Z^+$, we have $\chi(T^k) = \omega(T^k) = b_k(T)$.*

3 Chordal graphs and their powers

In this section we first consider an important subclass of chordal graphs consisting of interval graphs. We construct an interval representation for any power of an interval graph from the representation of the given interval graph. We then derive a new and elementary proof of the fact that any odd power of a chordal graph is again chordal. The approach of this proof is direct, and avoids the tedious case analysis in [1].

3.1 Powers of interval graphs

Assume an interval graph G is represented by real intervals $I_1, \ldots, I_n$, where $I_i = [a_i, b_i]$ for each $i \in \{1, \ldots, n\}$. For $k \in Z^+$ and $i \in \{1, \ldots, n\}$, we form a new interval $I_i(k) = [a_i, b_i(k)]$, where

$$b_i(k) = \max\{b_\alpha : d(I_i, I_\alpha) \leq k - 1\},$$

that is, the left endpoints of I_i and $I_i(k)$ are the same, but the right endpoint of $I_i(k)$ is the largest endpoint of an interval at a distance of at most $k - 1$ from I_i. Note that $d(I_i, I_\alpha) \leq k - 1$ means the existence of intervals $I_{l_1}, \ldots, I_{l_r}$, where $r \leq k - 1$, such that all the intersections $I_i \cap I_{l_1}, I_{l_1} \cap I_{l_2}, \ldots, I_{l_r} \cap I_\alpha$ are all nonempty. With this setup in mind, we obtain the following theorem.

Theorem 3.1 *If G is an interval graph represented by the real intervals $I_1, \ldots, I_n$, where $I_i = [a_i, b_i]$ for $1 \leq i \leq n$, and $k \in Z^+$, then G^k is represented by the intervals $I_1(k), \ldots, I_n(k)$.*

Proof. Let I_i and I_j be intervals in the representation of G of distance at most k from each other. We can assume $a_i < a_j$. Since $d(I_i, I_j) \leq k$, there is an interval I_l with

$$d(I_i, I_l) \leq k - 1 \quad \text{and} \quad d(I_l, I_j) \leq 1. \tag{2}$$

6

From (2) and the definition of $I_i(k)$, we get that $b_l \leq b_i(k)$. Again using (2), we see that $\max\{a_l, a_j\} < \min\{b_l, b_j\}$. Hence, we have

$$\max\{a_i, a_j\} = a_j \leq \max\{a_l, a_j\} < \min\{b_l, b_j\} \leq \min\{b_i(k), b_j(k)\}.$$

Therefore, $I_i(k) \cap I_j(k) \neq \emptyset$.

Suppose now, on the other hand, that $I_i(k) \cap I_j(k) \neq \emptyset$. Since we can continue to assume that $a_i < a_j$, this means that $a_j \in I_i(k)$. By definition of $I_i(k)$, there is an interval I_l with $d(I_i, I_l) \leq k - 1$ containing a_j, and hence $I_l \cap I_j \neq \emptyset$. Thus, $d(I_i, I_l) \leq k - 1$ and $d(I_l, I_j) \leq 1$ and so it follows that $d(I_i, I_j) \leq k$.

This shows $d(I_i, I_j) \leq k$ if and only if $I_i(k) \cap I_j(k) \neq \emptyset$, which completes the proof of our theorem. $\qquad\square$

Corollary 3.2 of Theorem 3.1 was previously proved in [18]. Raychaudhuri's proof was non-constructive using the characterization of interval graphs by Boland and Lekkerkerker from 1962 as precisely those chordal graphs that contain no *asteroidal triple* [3]. An asteroidal triple is a triple of vertices such that for any two of them, there is a path joining them that does not intersect the neighborhood of the third vertex.

Corollary 3.2 *If G is an interval graph, then so is G^k for any $k \in Z^+$.*

We now give an efficient $O(n \log k)$ algorithm for computing the interval representation of the power graph G^k of an interval graph G.

We first specify a linear-time algorithm, PRODUCT POWERS, to compute a *product* of two powers of interval graphs G^s and G^t, obtaining the power graph G^{s+t}. We assume as input a sorted list containing all the endpoints of G^s and G^t. The output is an interval representation of G^{s+t}, $[a_i, C[i]]$, where $C[i]$ is the larger of the i-th right endpoint of the interval with left endpoint a_i in G^s and the rightmost endpoint of an overlapping interval from G^t.

We process the endpoints of intervals of the graphs in decreasing order. We use a queue data structure Q to maintain the set of intervals from G^s that are live at any given time, and to report the one that reaches furthest to the right. In addition to the queue with its standard operations, we make use of a bit array indexed by the intervals of G^s.

ALGORITHM PRODUCT POWERS

```
Prod(G^s, G^t)
{ Input:  G^s with I_i  =  [a_i, b_i], for i  =  1, ..., n. }
{         G^t with I'_i  =  [a_i, d_i], for i  =  1, ..., n. }
{ Output:  Interval representation of G^{s+t}. }
```

{ The $3n$ endpoints, the duplicate left endpoints are not included, }
{ of the intervals are given in sorted order. In the sorting, ties }
{ are broken arbitrarily. }
{ interval[i] is the index of the interval with the i-th left endpoint. }
{ type[i] is an entry denoting the type of endpoint. }
{ interval[i] and type[i] have $3n$ entries. }
{ Head(Q) = null = 0, if Q is empty. }
{ Right$_l(\cdot)$ is a function that takes an interval index and returns }
{ the right endpoint of the interval in G^l, where $l = s$ or t. }

```
for i ← 0 to n do
   marked[i]←0;
for i ← 3n downto 1 do
   if (type[i] is a left endpoint) then
      marked[interval[i]]←1;
   if (type[i] is a right endpoint from Gˢ) then
      Enqueue(Q,interval[i]);
   if (type[i] is a right endpoint from Gᵗ) then
      while (marked[Head(Q)]) do
         Dequeue(Q);
      if Head(Q) ≠ null then
         C[interval[i]]← Rightₛ(Head(Q));
      else
         C[interval[i]]← Rightₜ(interval[i]);
for i ← 1 to n do
   output intervals [aᵢ, C[i]];
```

We use a traditional trick to efficiently compute the power graph G^k. To form the square graph G^2, we compute $\texttt{Prod}(G,G)$, and to compute G^{2^t}, we compute the square t times. To compute an arbitrary power, we then form the product of the powers corresponding to the bit representation of k, in at most $2\log k$ applications of $\texttt{Prod}$. For example, notice that $G^{10} = (((G^2)^2)^2 \cdot G^2$.

A straightforward application of our approach yields the same result for circular-arc graphs. A *circular-arc graph* is a graph whose vertex set can be represented by a collection of proper, closed arcs of the unit circle in the real plane, where two vertices are connected by an edge if and only if the corresponding arcs have a nonempty intersection. Note that every interval graph is, in particular, a circular-arc graph. Also notice that every cycle, of arbitrary length, is a circular-arc graph. Therefore, in general, circular-arc graphs are not chordal graphs. In modifying the definitions of $b_i(k)$ and $I_i(k)$ in the obvious way, we obtain the following corollary that is analogous to Corollary 3.2 for interval graphs.

Corollary 3.3 *If G is a circular-arc graph, represented by arcs $I_1, \ldots, I_n$, and $k \in Z^+$, then G^k is a circular-arc graph represented by the arcs $I_1(k), \ldots, I_n(k)$.*

3.2 Odd powers of chordal graphs

It has been pointed out numerous times in the literature that squares of chordal graphs are not necessarily chordal, and a classic example of such a graph is given in Figure 1, (see [15], [16], and [19] for the same example). In fact, we will see more generally in the next section that the subclass of the chordal graphs consisting of split graphs are such that for *any* given graph, there is a corresponding split graph, whose square has the given graph as an induced subgraph.
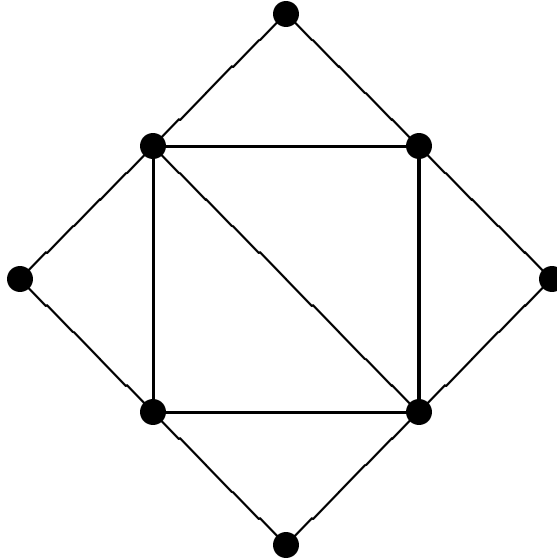

Figure 1: An example of a chordal graph whose square is not chordal. In the square of this graph, only two edges are missing from K_8. They are the "wide horizontal edge" and the "tall vertical edge."

We will conclude this section with a few lemmas and an observation. They will enable us to give a direct and elementary proof of the fact that any odd power of a chordal graph is again chordal. This result was previously proved in [1] using exhaustive case analysis.

A graph is *outerplanar* if it has a planar embedding such that every vertex lies on the unbounded face. We call an outerplanar graph G on $n \geq 4$ vertices *fully triangulated* if there is a planar embedding G^* of G, which has one region bounded by n edges, and all of the other regions bounded

by three edges. In other words, G is fully triangulated if G consists of a cycle, whose interior face is divided into triangles. An induction on $n \geq 3$, yields the following lemma.

Lemma 3.4 *If C is a simple m-cycle in a chordal graph G, then the induced subgraph $G[C]$, generated by C in G, contains a fully triangulated outerplanar graph on the m vertices of the cycle.*

For simple contractions G/e we obtain the following lemma.

Lemma 3.5 *Let G be a fully triangulated outerplanar graph on four or more vertices. If e is an edge bounding the non-triangular region in one (and hence every) planar embedding of G, then the graph G/e is also a fully triangulated outerplanar graph.*

Proof. Clearly, G/e is outerplanar. If G has n vertices, then every planar embedding of G/e has $n-4$ triangular regions and one region bounded by $n-1$ edges. Hence, it is fully triangulated. $\square$

The proof of the next lemma is straightforward.

Lemma 3.6 *Let u_1, u_2, u_3, and u_4 be distinct vertices in a graph G. For $i \in \{1,3\}$, assume $p(u_i, u_{i+1})$ is a simple path of length k or less from u_i to u_{i+1}. If $p(u_1, u_2)$ and $p(u_3, u_4)$ have any vertex in common, then either $d_G(u_1, u_3) \leq k$ or $d_G(u_2, u_4) \leq k$.*

For vertices u and v in a graph, $p(u, v)$ denotes a path between u and v. The notation $l_p(u, v)$ denotes the length of $p(u, v)$. We sometimes sub- or superscript the name of a path. If there is no danger of ambiguity, the length of a labeled path $p^*(u, v)$ will be denoted by the same label $l^*(u, v)$, instead of $l_{p^*}(u, v)$.

Observation 3.7 *Let u, v, and w be three distinct vertices in a connected graph. Let $p(u, v)$ ($p(u, w)$) be a path of shortest length from u to v (respectively, from u to w). Then there exists a unique vertex u' and a partition of the paths*

$$p(u, v) = p_v(u, u') \cup p_v(u', v)$$
$$p(u, w) = p_w(u, u') \cup p_w(u', w),$$

where $p_v(u', v)$ and $p_w(u', w)$ are vertex disjoint, except for their initial vertex u', and $l_v(u, u') = l_w(u, u')$.

We now present a different and more direct proof from those appearing in [1] and [7] of the following theorem:

Theorem 3.8 *Let $k \in Z^+$ be an odd integer. If G is a chordal graph, then G^k is also chordal.*

Proof. Consider a simple m-cycle in G^k. This m-cycle corresponds to distinct vertices $u_1, \ldots, u_m$ in G, and m simple paths in G, each of length k or less, namely $p(u_1, u_2), \ldots, p(u_{m-1}, u_m)$ and $p(u_m, u_1)$, connecting the vertices cyclically. Throughout the proof, let u_{m+1} (u'_{m+1}) equal u_1 (respectively, u'_1). We assume further that all these paths have minimal lengths, that is, $l_p(u_i, u_{i+1}) = d_G(u_i, u_{i+1})$ for all i.

Let $i \in [m]$. The paths $p(u_{i-1}, u_i)$ and $p(u_i, u_{i+1})$ give rise to a unique vertex u'_i satisfying Observation 3.7. Similarly, the paths $p(u_i, u_{i+1})$ and $p(u_{i+1}, u_{i+2})$ give rise to a unique vertex u'_{i+1}. We see by Lemma 3.6 that the vertices u'_i and u'_{i+1} on $p(u_i, u_{i+1})$ must be distinct, and appear in the same order on $p(u_i, u_{i+1})$ as u_i and u_{i+1} do. Hence, the m-cycle $C = C(u'_1, \ldots, u'_m)$ is simple, yielding disjoint paths $p^*(u'_1, u'_2), \ldots, p^*(u'_{m-1}, u'_m)$ and $p^*(u'_m, u'_1)$.

For each $i \in [m]$ pick one of the paths from u_i to u'_i (along $p(u_{i-1}, u_i)$ or $p(u_i, u_{i+1})$) and call it $p^*(u_i, u'_i)$.

We have now shown that we can assume the vertices $u_1, \ldots, u_m$ are connected in an "octopus"-like manner by the simple m-cycle C, together with paths, or "tentacles" $p^*(u_i, u'_i)$, which are disjoint among themselves and from C. We call the entire octopus graph C^*.

By Lemma 3.4, G contains a fully triangulated outerplanar graph on $u'_1, \ldots, u'_m$. Note that

$$l^*(u_i, u'_i) + l^*(u'_i, u'_{i+1}) + l^*(u'_{i+1}, u_{i+1}) = d_G(u_i, u_{i+1}) \leq k$$

for each $i \in [m]$.

Since k is odd, then for each $i \in [m]$ there is an edge $\{y_i, x_{i+1}\}$ on $p^*(u'_i, u'_{i+1})$, in this order, such that for any vertex z on the arc of C defined by $\{x_i, u'_i, y_i\}$, the path from z to u_i along this arc on C and the "tentacle" $p^*(u'_i, u_i)$ has length at most $(k-1)/2$.

Denote by T_i the subtree of C^* connecting u_i, x_i, and y_i. We note that T_i has at most three leaves, and they are among $\{u_i, x_i, y_i\}$. Moreover, at most one possible vertex, u'_i, of T_i has degree three. We now have a partition

$$V(C^*) = V(T_1) \cup \cdots \cup V(T_m).$$

Contracting each T_i to a single vertex t_i will give a graph, which by Lemmas 3.4 and 3.5 will contain a fully triangulated outerplanar graph C^{**} on $t_1, \ldots, t_m$ as a subgraph (contracting each "tentacle," $p^*(u_i, u'_i)$, has no effect on the full triangularity of C^{**}). Therefore, there are two vertices t_i and t_j, with neither $j = i + 1$ nor $i = j + 1$, connected by an edge in C^{**}. Hence, there must be an edge in C^* with one endvertex in T_i and the other

in T_j, more specifically, an edge between z_i and z_j, where z_i is a vertex in T_i and where z_j is a vertex in T_j. We conclude that

$$
\begin{aligned}
d_G(u_i, u_j) & \leq & d_G(u_i, z_i) + d_G(z_i, z_j) + d_G(z_j, u_j) \\
& \leq & l^*(u_i, z_i) + 1 + l^*(z_j, u_j) \\
& \leq & \frac{k-1}{2} + 1 + \frac{k-1}{2} \\
& = & k.
\end{aligned}
$$

Hence, there is an edge connecting u_i to u_j in G^k, thus completing the proof of our theorem. □

4 Approximate coloring of powers of graphs

We have seen that odd powers of chordal graphs are easy to color, since they are also chordal, while even powers are generally not chordal. In the current section we shall study how difficult the even powers of chordal graphs are to color. In some respects, the issue is how far from being chordal these power graphs are. We measure the difficulty in terms of how good an estimate of the chromatic number an effective algorithm can find.

The coloring problem is $\rho(n)$-*approximable* on a given class of graphs if, there exists a polynomial time algorithm that for each graph G on n vertices outputs a coloring with at most $\rho(n)\chi(G)$. We say that a problem is *hard to approximate* within a given factor, if the contrary would yield the conclusion that $NP \neq ZPP$, the class of problems with polynomial-time zero-error randomized algorithms.

The main result of this section is that coloring the even powers of chordal graphs is hard to approximate within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$. In fact, coloring squares of split graphs is hard within that factor. Interestingly, this can be matched with a simple $O(\sqrt{n})$-approximation algorithm for even powers of arbitrary graphs. We then consider odd powers of general graphs, and give nearly matching bounds on their approximability.

For a graph G, let $\alpha(G)$ denote the size of a maximum independent set in G. That is, $\alpha(G)$ gives the *independence number* of G.

First, we give an approximation lower bound for coloring even powers of chordal graphs.

Theorem 4.1 *The problem of coloring squares of split graphs is hard to approximate within $O(n^{\frac{1}{2}-\epsilon})$ for any $\epsilon > 0$.*

12

Proof. We give a reduction from Graph Coloring which is known to be hard to approximate within an $n^{1-\epsilon}$ factor [10].

Given a graph G on N vertices, we construct a graph H that contains N copies of each vertex in G along with an additional clique on N vertices. Let $n = N^2 + N$ denote the number of vertices in H. A copy of vertex v_i is adjacent to the j-th clique vertex if and only if $\{v_i, v_j\}$ is an edge in G or if $i = j$. Formally, let

$$
\begin{aligned}
V(H) &= \{x_i, u_{i,j} : 1 \le i, j \le N\}, \text{ and} \\
E(H) &= \{\{x_i, u_{j,l}\} : \{v_i, v_j\} \in E(G) \text{ or } i = j\} \\
&\quad \cup \{\{x_i, x_j\} : 1 \le i, j \le N\}.
\end{aligned}
$$

In H^2, the graph induced by $\{u_{j,l} : j = 1, 2, \ldots, N\}$ is a copy of G, for each $l = 1, \ldots, N$. The vertices x_i are adjacent to every vertex in H^2. Copies of a vertex v are adjacent to precisely the copies of v's neighbors in G.

The result of Feige and Kilian [10] states that it is hard to distinguish between two cases: (i) $\chi(G) \le N^\epsilon$, and (ii) $\alpha(G) \le N^{1-\epsilon}$ for any fixed $\epsilon > 0$. Other cases from [10] are not relevant to our work.

Observe that if I is an independent set in H^2, then it consists of copies of distinct vertices that form an independent set in G. Therefore, $\alpha(H^2) \le \alpha(G)$. Thus, if $\alpha(G) \le N^\epsilon$, then $\alpha(H^2) \le N^\epsilon$. Also, we have $\chi(H^2) \ge (N^2 + N)/\alpha(H^2) \ge N^{2-\epsilon}$.

Also, trivially $\chi(H^2) \le \chi(G) \cdot N + N$. Thus, if $\chi(G) \le N^\epsilon$, then $\chi(H^2) \le N^{1+\epsilon} + N$. So, if we could distinguish between the cases when $\chi(H^2) \le N^{1+\epsilon} + N$ and $\chi(H^2) \ge N^{2-\epsilon}$, we could distinguish between the two cases of the result of [10]. Hence, it is hard to approximate the chromatic number of the square graph H^2 within a factor of $N^{1-2\epsilon}$ which is $n^{1/2-\epsilon}$ if we ignore lower order terms. $\square$

The construction of Theorem 4.1 can be modified to give the same hardness result for another subclass of perfect graphs: bipartite graphs. Simply remove all edges between x-vertices.

Cubes, and thus all higher powers, of split graphs are already complete graphs. However, chordal graphs remain hard to color for larger even powers.

Theorem 4.2 *Coloring even powers of chordal graphs is hard to approximate within $n^{1/2-\epsilon}$ for any $\epsilon > 0$.*

Proof. Let $k = 2t$ and assume $t \ge 2$. We modify the construction of Theorem 4.1. Between each vertex x_i and the corresponding $u_{i,x}$ vertices, we add a path of $t - 1$ vertices.

$$
V(H) = \{x_i, y_{i,1}, y_{i,2}, \ldots, y_{i,t-1}, u_{i,l}, : 1 \le i, l \le N\}, \text{ and}
$$

13

$$\begin{aligned}
E(H) \;=\; & \{\{x_j, y_{i,1}\} : \{v_i, v_j\} \in E(G) \text{ or } i = j\} \\
& \cup\{\{x_i, x_j\} : 1 \le i, j \le N\} \\
& \cup\{\{y_{i,1}, y_{i,2}\}, \ldots, \{y_{i,t-2}, y_{i,t-1}\}, \\
& \quad\;\; \{y_{i,t-1}, u_{i,l}\} : 1 \le l, i \le N\}
\end{aligned}$$

The graph H contains $N^2 + tN$ vertices. The x and y-vertices form a clique on tN vertices. The subgraph of H^k on the $u_{i,j}$ vertices is the same as in Theorem 4.1. Again, $\alpha(H^k) \le \alpha(G)$, so $\chi(H^k) \ge (N^2 + tN)/\alpha(G)$, while $\chi(H^k) \le (\chi(G) + t) \cdot N$. The theorem now follows by the same arguments for any $t = O(N^\epsilon)$. $\qquad\square$

We note that the *NP*-hardness reduction of Lin and Skiena [16] yields nearly the same result for general graphs, or an $(n/k)^{1/2-\epsilon}$-hardness. We can give a simple matching upper bound that holds for arbitrary graphs G.

Theorem 4.3 *Coloring even powers of graphs is $O(\sqrt{n})$-approximable by a simple greedy algorithm.*

Proof. If suffices to show this for square graphs since G^{2t} is the square of the graph G^t.

The maximum degree $\Delta(G^2)$ of G^2 is at most $\Delta(G)^2$. Thus, a first-fit greedy algorithm uses at most $\min\{\Delta(G)^2 + 1, n\}$ colors on G^2. Any neighborhood in G forms a clique in G^2. Therefore, the optimal solution requires at least $\Delta(G) + 1$ colors. Hence, we get a performance ratio which is at most $\min\{\Delta(G), n/\Delta(G)\} \le \sqrt{n}$. $\qquad\square$

Another measure of non-chordality would be the types of graphs contained as subgraphs. The proofs of Theorems 4.1 and 4.2 answer that.

Observation 4.4 *The square of a split graph can contain an arbitrary graph as a subgraph.*

4.1 Odd powers of general graphs

It may be fruitful to study the odd powers of general graphs. For general graphs, the $\sqrt{n}$-hardness result also holds for odd powers.

Theorem 4.5 *Coloring odd powers of graphs is hard to approximate within $n^{1/2-\epsilon}$ for any $\epsilon > 0$.*

Proof. For the case of $k = 2t$, with t greater than 1, we construct the following graph H on $tN + N^2$ vertices, when given a graph G on N vertices.

14

The graph consists of G, a path of $t-1$ vertices attached to each vertex of G, and a set of N vertices attached to the end node of each path. Formally,

$$
\begin{aligned}
V(H) &= \{v_i, y_{i,1}, y_{i,2}, \ldots, y_{i,t-1}, u_{i,j}, : 1 \le i, j \le N\}, \text{ and} \\
E(H) &= \{\{v_i, v_j\} : \{v_i, v_j\} \in E(G)\} \\
&\quad \cup \{\{v_1, y_{i,1}\}, \{y_{i,1}, y_{i,2}\}, \ldots, \{y_{i,t-2}, y_{i,t-1}\}, \\
&\qquad \{y_{i,t-1}, u_{i,l}\} : 1 \le l, i \le N\}
\end{aligned}
$$

The u-vertices induce in H^k the same subgraph as in Theorems 4.1 and 4.2. The theorem now follows by the same argument. □

On the positive side, just as the odd integers play an important role in Theorem 3.8, we can obtain a nontrivial approximation for vertex coloring all odd powers of graphs.

Theorem 4.6 *Coloring odd powers of graphs is $O(n^{2/3})$-approximable.*

Proof. Let $k = 2t - 1$. Let D_i be the maximum over all vertices v of the number of vertices within distance i from v. The maximum degree of G^k is D_k. The first-fit algorithm then uses at most $D_k + 1$ colors. Vertices within distance $t - 1$ from a given node v must form a clique in G^k. Thus, the clique and the chromatic number of G^k is at least $D_{t-1} + 1$. Hence, the performance ratio is at most $\min\{D_k/D_{t-1}, n/D_{t-1}\}$. Clearly, $D_k \le (D_{t-1})^3$. Thus, the performance ratio is at most $n^{2/3}$. Note, that we can also bound the performance ratio by opt^2, where opt is the size of the optimal solution. □

This yields an interesting comparison with the Independent Set problem. It was shown in [12] that independent sets in odd powers of graphs are hard to approximate within an $n^{1-\epsilon}$ factor. This yields the first nontrivial separation between approximations of Graph Coloring and Independent Set in a natural class of graphs.

Corollary 4.7 *The approximability of Independent Set and Graph Coloring in odd powers of graphs differs by a factor of at least $n^{1/6-\epsilon}$ for any $\epsilon > 0$.*

5   Chromatic polynomials for powers of trees

In this last section we further study powers of trees. We will give two exact and closed formulas for the chromatic polynomial of the k-th power of a tree on n vertices. The *chromatic polynomial* of a graph G, denoted $\chi_G(t)$, specifies how many different ways there are of coloring G using $t \ge \chi(G)$

colors. Note that both $G \setminus e$ and G/e have fewer edges than G. Hence, the fact that $\chi_G(t)$ is actually a polynomial in t can easily be verified by using induction on $|E(G)|$, and the recurrence

$$\chi_G(t) = \chi_{G \setminus e}(t) - \chi_{G/e}(t)$$

(see [20, Theorem 5.3.4, page 195]). For example, if K_n denotes the complete graph on n vertices, then $\chi_{K_n}(t) = t(t-1)\cdots(t-n+1)$ for t greater than or equal to n. Likewise, for any tree T on n vertices, we have $\chi_T(t) = t(t-1)^{n-1}$. Moreover, if G is a graph on n vertices and $\chi_G(t) = t(t-1)^{n-1}$, then one can show by an easy induction that G is indeed a tree on n vertices.

Let G be a chordal graph. The simplicial elimination ordering, $\{v_1, \ldots, v_n\}$ on $V(G)$, yields that the chromatic polynomial $\chi_G(t)$ of G has the following form

$$\chi_G(t) = \prod_{i=1}^{n}(t - d(i)), \tag{3}$$

where $d(i) = |N(v_i) \cap \{v_1, \ldots, v_{i-1}\}|$. Hence, all the roots of the chromatic polynomial for chordal graphs are nonnegative integers. Note that in this case, the chromatic number of G is given by $\chi(G) = r + 1$, where $r = \max_i\{d(i)\}$, the largest root of $\chi_G(t)$.

Since a fully triangulated outerplanar graph is chordal, the following corollary is immediate from Theorem 3.8.

Corollary 5.1 *If G is a fully triangulated outerplanar graph and $k \in Z^+$ is an odd integer, then the chromatic polynomial for G^k has only nonnegative integers as roots.*

Since every power of a tree is chordal, we know that all the roots of $\chi_{T^k}(t)$ are nonnegative integers. In order to derive the formulas for $\chi_{T^k}(t)$, we need to formulate some key ideas. Recall the meaning of k-ball presented in Definition 2.1.

Clearly, any intersection of k-balls in T is again a k-ball. For a tree T and $k \in Z^+$, let $B_1, \ldots, B_m$ be the complete listing of all the k-balls of T. For the remainder of this paper, for $S \subseteq [m]$ let

$$f_S(t) = t(t-1)\cdots(t - b_S + 1),$$

be the *falling factorial function*, where $b_S = \left|\bigcap_{s \in S} B_s\right|$ (sometimes in the literature, this function is denoted by $(t)_{b_S}$). If $b_S = 0$, then let $f_S(t) = 1$. The function $f_S(t)$ will play an important role in what follows.

16

5.1 Formulas for chromatic polynomials

We now present the first formula for $\chi_{T^k}(t)$.

Theorem 5.2 *Let T be a tree, $k \in Z^+$, and $B_1, \ldots, B_m$ all of T's k-balls. We have*

$$\chi_{T^k}(t) = \prod_{S \subseteq [m]} f_S(t)^{(-1)^{|S|-1}}.$$

Proof. We will use induction on $n = |V(T)|$. If T has exactly one vertex, then $\chi_{T^k}(t) = t$, which agrees with the formula.

Suppose T has $n \geq 2$ vertices. Let $u \in V(T)$ be an endvertex of a longest path in T. By Lemma 2.2 all of T's vertices, which are of distance k or less from u, are contained in a k-ball of T.

Let $B_1, \ldots, B_m$ be all the k-balls of T, enumerated in such a way that

$$u \in B_1, \ldots, B_l \text{ and}$$
$$u \notin B_{l+1}, \ldots, B_m,$$

and such that $|B_1| \leq \cdots \leq |B_l|$. Again, by Lemma 2.2, we may assume $B_i \subseteq B_l$ for all $i \in [l]$, that is, B_l is precisely the k-ball of T consisting of u and all the vertices of distance k or less from u in T. Lastly, since $B_l \setminus \{u\}$ is a k-ball in T, we can assume $B_{l+1} = B_l \setminus \{u\}$ and $B_{l+1} \neq \emptyset$. Consider the tree $T_1 = T \setminus \{u\}$ on $n - 1$ vertices. By our choice of the vertex u, we have by the definition of chromatic polynomials that

$$\chi_{T^k}(t) = (t - |B_l| + 1)\chi_{T_1^k}(t).$$

Note that $B_{l+1}, \ldots, B_m$ are precisely all the k-balls of T_1, so by the induction hypothesis we have

$$\chi_{T_1^k}(t) = \prod_{R \subseteq [l+1;m]} f_R(t)^{(-1)^{|R|-1}}.$$

Let $\mathbf{S} = \{S \subseteq [m] : S \nsubseteq [l+1;m]\}$. It therefore suffices to show that

$$\prod_{S \subseteq \mathbf{S}} f_S(t)^{(-1)^{|S|-1}} = t - |B_l| + 1. \tag{4}$$

Now, $\mathbf{S}$ consists of those S that contain at least one element of $[l]$. Hence, $\bigcap_{s \in \mathbf{S}} B_s$ is a k-ball contained in one of the k-balls $B_1, \ldots, B_l$. Define

$$\mathbf{S}' = \{S \in \mathbf{S} : S \cap [l] \neq \{l\}\}.$$

We have a partition $\mathbf{S}' = \mathbf{S}'_1 \cup \mathbf{S}'_2$, where $\mathbf{S}'_1 \subseteq \mathbf{S}'$ is the set of those $S \in \mathbf{S}'$ not containing l, and $\mathbf{S}'_2 \subseteq \mathbf{S}'$ is the set of those $S \in \mathbf{S}'$ that do contain l. There is a one-to-one correspondence between $\mathbf{S}'_1$ and $\mathbf{S}'_2$ given by

$$\mathbf{S}'_1 \ni S' \quad \leftrightarrow \quad S' \cup \{l\} \in \mathbf{S}'_2.$$

17

For $S' \in \mathbf{S}'_1$, we have $f_{S'}(t) = f_{S' \cup \{l\}}(t)$. Thus,

$$f_{S'}(t)^{(-1)^{|S'|-1}} \cdot f_{S' \cup \{l\}}(t)^{(-1)^{|S' \cup \{l\}|-1}} = 1.$$

Therefore, it suffices to consider only those $S \in \mathbf{S}$ that contain l and no other element of $[l]$. So,

$$\prod_{S \in \mathbf{S}} f_S(t)^{(-1)^{|S|-1}} = \prod_{S'' \subseteq [l+1;m]} f_{S'' \cup \{l\}}(t)^{(-1)^{|S''|}}.$$

For a nonempty $S'' \subseteq [l+1;m]$ either $l+1 \in S''$ or not. Let

$$\mathbf{S}'' = \{S'' \cup \{l\} : S'' \subseteq [l+1;m]\}.$$

Again, we get a partition $\mathbf{S}'' = \mathbf{S}''_1 \cup \mathbf{S}''_2$, where $\mathbf{S}''_1$ is the set of elements of $\mathbf{S}''$ that do not contain $l+1$, and $\mathbf{S}''_2$ is the set of those elements of $\mathbf{S}''$ that do contain $l+1$. We also have a one-to-one correspondence

$$\mathbf{S}''_1 \ni S'' \cup \{l\} \quad \leftrightarrow \quad S'' \cup \{l\} \cup \{l+1\} \in \mathbf{S}''_2,$$

where $S'' \subseteq [l+2;m]$. Since $B_{l+1} = B_l \setminus \{u\}$ for a nonempty such S'', we have that

$$\bigcap_{s \in S'' \cup \{l\}} B_s = \bigcap_{s \in S'' \cup \{l\} \cup \{l+1\}} B_s.$$

Hence, if we put $S_l = S'' \cup \{l\}$ and $S_{l+} = S'' \cup \{l\} \cup \{l+1\}$, then we have

$$f_{S_l}(t)^{(-1)^{|S_l|-1}} \cdot f_{S_{l+}}(t)^{(-1)^{|S_{l+}|-1}} = 1.$$

For $S'' = \emptyset$, we obtain

$$f_{\{l\}}(t) \cdot f_{\{l,l+1\}}(t)^{-1} = t - |B_l| + 1.$$

This proves that (4) is correct, and hence our theorem. $\qquad\square$

REMARK: Clearly the product displayed in Theorem 5.2 is large (in fact, a product of 2^m factors), and it does not give an efficient way to calculate the chromatic polynomial for T^k. A considerable simplification appears in the following theorem, which is the second exact formula for $\chi_{T^k}(t)$.

Theorem 5.3 *Let T be a tree and $k \in Z^+$. If $B_1, \ldots, B_c$ are all the distinct maximal k-balls of T, then*

$$\chi_{T^k}(t) = \prod_{S \subseteq [c]} f_S(t)^{(-1)^{|S|-1}}.$$

18

Proof. We will use induction on $n = |V(T)|$. The formula clearly holds when n equals 1.

Let $B_1, \ldots, B_c$ be all the maximal k-balls of T. As in the proof of Theorem 5.2, let $u \in V(T)$ be a leaf of T that is an endvertex of a longest path in T. Let $T_1 = T \setminus \{u\}$. By Lemma 2.2, we may assume that B_c is the (unique!) maximal k-ball containing u. Indeed by the definition of chromatic polynomials, we have

$$\chi_{T^k}(t) = (t - |B_c| + 1)\chi_{T_1^k}(t).$$

We must consider two possibilities: whether $B_c \setminus \{u\}$ is a maximal k-ball of T_1 or not.

CASE ONE: Suppose $B_c \setminus \{u\}$ is a maximal k-ball of T_1. So, the maximal k-balls of T_1 are $B_1, \ldots, B_{c-1}$, and $B_c \setminus \{u\}$. Let

$$B_i^u = \left\{ \begin{array}{ll} B_i & \text{if } i \leq c - 1 \\ B_c \setminus \{u\} & \text{if } i = c. \end{array} \right.$$

Let $f_S^u(t) = t(t-1)\cdots(t - b_S^u + 1)$, where $b_S^u = |\bigcap_{s \in S} B_s^u|$. We consider two subcases depending on whether $S = \{c\}$ or not. Note that for $S \neq \{c\}$, we have

$$\bigcap_{s \in S} B_s^u = \bigcap_{s \in S} B_s.$$

Thus,

$$f_S(t) = f_S^u(t)$$

When $S = \{c\}$, we have

$$\begin{array}{rcl} f_S(t) & = & f_{\{c\}}(t) \\ & = & (t - |B_c| + 1)f_{\{c\}}^u(t) \\ & = & (t - |B_c| + 1)f_S^u(t). \end{array}$$

Therefore, using the induction hypothesis, we have

$$\begin{array}{rcl} \chi_{T^k}(t) & = & (t - |B_c| + 1)\chi_{T_1^k}(t) \\[2mm] & = & (t - |B_c| + 1) \cdot \prod_{S \subseteq [c]} f_S^u(t)^{(-1)^{|S|-1}} \\[2mm] & = & (t - |B_c| + 1)f_{\{c\}}^u(t) \cdot \prod_{S \subseteq [c], S \notin \{\{c\}\}} f_S^u(t)^{(-1)^{|S|-1}} \\[2mm] & = & f_{\{c\}}(t) \cdot \prod_{S \subseteq [c], S \notin \{\{c\}\}} f_S(t)^{(-1)^{|S|-1}} \\[2mm] & = & \prod_{S \subseteq [c]} f_S(t)^{(-1)^{|S|-1}}. \end{array}$$

19

This completes the induction in this case.

Case Two: Suppose $B_c \setminus \{u\}$ is not a maximal k-ball in T_1. Since $B_1, \ldots, B_{c-1}$ are all maximal in T, they are all maximal in T_1. We know that $B_c \setminus \{u\}$ is contained in some maximal k-ball of T_1. This maximal k-ball must also be maximal in T; and therefore, we can assume $B_c \setminus \{u\} \subseteq B_{c-1}$, so $B_1, \ldots, B_{c-1}$ are precisely all the maximal k-balls in T_1. Hence, we have by the induction hypothesis

$$\chi_{T_1^k}(t) = \prod_{S \subseteq [c-1]} f_S(t)^{(-1)^{|S|-1}}.$$

It suffices to show that

$$\prod_{S \subseteq [c], S \not\subseteq [c-1]} f_S(t)^{(-1)^{|S|-1}} = t - |B_c| + 1. \tag{5}$$

If $\mathbf{S} = \{S \subseteq [c] : S \not\subseteq [c-1]\}$, then $\mathbf{S} = \{S' \cup \{c\} : S' \subseteq [c-1]\}$. Note that $B_c \cap B_{c-1} = B_c \setminus \{u\}$ and also that $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$, where

$$\mathbf{S}_1 = \{S'' \cup \{c\} : S'' \subseteq [c-2]\},$$

and

$$\mathbf{S}_2 = \{S'' \cup \{c-1\} \cup \{c\} : S'' \subseteq [c-2]\}.$$

For a nonempty set $S'' \subseteq [c-2]$, we have

$$\bigcap_{s \in S'' \cup \{c\}} B_s = \bigcap_{s \in S'' \cup \{c-1\} \cup \{c\}} B_s.$$

Therefore, if $S_c = S'' \cup \{c\}$ and $S_{c-} = S'' \cup \{c-1\} \cup \{c\}$, then

$$f_{S_c}(t)^{(-1)^{|S_c|-1}} \cdot f_{S_{c-}}(t)^{(-1)^{|S_{c-}|-1}} = 1.$$

Also, $f_{\{c\}}(t) \cdot f_{\{c-1,c\}}(t)^{-1} = t - |B_c| + 1$. This shows that (5) is valid and so completes the induction. □

Remark: Although the formula in Theorem 5.3 is a substantial simplification of the one given in Theorem 5.2, it still does not yield a fast way to calculate $\chi_{T^k}(t)$ for a given integer t.

For the second power of T, we have the following corollary to Theorem 5.3.

Corollary 5.4 *Let T be a tree; $d_1, \ldots, d_m$ the degrees of the m non-leaves of T; $d = \max_i\{d_i\}$; and for each $i \in [m]$ let $\alpha_i = |\{j \in [m] : d_j \geq i\}|$. Then*

$$\chi_{T^2}(t) = t(t-1) \prod_{i=2}^{d} (t-i)^{\alpha_i}.$$

20

Proof. The maximal 2-balls are the vertices of the sub-stars of T centered at the non-leaves of T. An intersection of 2-balls has one of the following forms:

- a 2-ball

- a 1-ball (that is, a set of two neighboring vertices neither of which are leaves)

- a 0-ball (that is, a single non-leaf vertex)

- empty

We consider the contribution of each type of intersection to the overall product in turn. If $B_1, \ldots, B_m$ are all the maximal 2-balls, then

$$N = \prod_{i \in [m]} f_{\{i\}}(t) = \prod_{i=1}^{m} t(t-1) \cdots (t-d_i)$$

is the product corresponding to intersections forming 2-balls.

The product corresponding to all the 1-balls is

$$D = \prod_{i=1}^{m-1} t(t-1),$$

since each 1-ball is an intersection of exactly two 2-balls of T, and there are exactly $m - 1$ such 1-balls in T, which are not connected to leaves.

Next we consider the contribution due to intersections that are 0-balls. Such an intersection is a single vertex of T that is neither a leaf in T nor a leaf in the tree T', where $T' = T \setminus \{\text{leaves of } T\}$. Assuming $u \in V(T)$ is such a 0-ball, then

$$\{u\} = B_{i_1} \cap \cdots \cap B_{i_\gamma},$$

where $\gamma \geq 2$. If u is not the center of any of the 2-balls $B_{i_1}, \ldots, B_{i_\gamma}$, then u is the center of some other 2-ball, say B_j, where $j \notin \{i_1, \ldots, i_\gamma\}$. In this case

$$\{u\} = B_{i_1} \cap \cdots \cap B_{i_\gamma} \cap B_j$$

also, and hence we have

$$f_{\{i_1, \ldots, i_\gamma, j\}}(t)^{(-1)^\gamma} \cdot f_{\{i_1, \ldots, i_\gamma\}}(t)^{(-1)^{\gamma-1}} = 1.$$

Likewise, if u is the center of one of the 2-balls, say B_{i_1}, then

$$\{u\} = B_{i_2} \cap \cdots \cap B_{i_\gamma},$$

and therefore,

$$f_{\{i_2,\ldots,i_\gamma\}}(t)^{(-1)^{\gamma-1}} \cdot f_{\{i_1,\ldots,i_\gamma\}}(t)^{(-1)^{\gamma-2}} = 1.$$

We conclude that the polynomial product corresponding to all the 0-balls is 1, and hence by Theorem 5.3

$$\chi_{T^2}(t) = \frac{N}{D} = t(t-1) \prod_{i=1}^{m} (t-2) \cdots (t-d_i).$$

Counting the exponents of each of the factors $(t-i)$ proves our corollary. $\square$

REMARK: A simple proof of Corollary 5.4 can be obtained by using the same inductive idea as in the proofs of Theorems 5.2 and 5.3.

5.2 Algorithm for chromatic polynomial evaluation

Given an integer t, the chromatic polynomial $\chi_{T^k}(t)$ can actually be evaluated in linear time for each fixed power k. In what follows, we present an $O(kn)$ algorithm to evaluate $\chi_{T^k}(t)$, where T has n vertices and t is any integer. To evaluate this polynomial it suffices to get a simplicial elimination ordering of the vertices of T^k, together with the values $d_k(1),\ldots,d_k(n)$, to yield $\chi_{T^k}(t)$ as a product as in (3).

The next observation follows by an inductive application of Lemma 2.2. Any root of the tree T is sufficient.

Observation 5.5 *Breadth-First-Search (BFS) order of a tree T gives a strong elimination ordering of the power graph T^k.*

Proof. The BFS ordering, rooted at any fixed node, divides the nodes into levels, with the lowest leaves at level 0. Consider a leaf u at level 0 and let $N_k(u)$ be its neighborhood in T^k. Let v and w be vertices in $N_k(u)$ at levels i and j, respectively, with $i \leq j$. Then, we claim that $N_k(v) \subseteq N_k(w)$. To see this, let q be the least common ancestor of v and w. $N_k(w)$ contains all descendants of q, since their distance to w is at most the larger of the distances of w to u and v to u. Hence, u is strongly simplicial in T^k.

Observe that for a connected subtree S of T, the power graph S^k is the same as the subgraph of T^k induced by the vertices of S. Hence, the same argument applies by induction to the tree $T \setminus u$, yielding a strong elimination ordering of T^k. $\square$

Below we give an algorithm, CHROMATIC POLYNOMIAL, for computing a representation of the chromatic polynomial. Namely, we find the values $d_k(1),\ldots,d_k(n)$, where $d_k(i) = |N_k(v_i) \cap \{v_1,\ldots,v_{i-1}\}|$. Here $N_k(i)$

denotes the neighborhood of v_i in T^k. For each node v we maintain its number of descendants of distance t, where $t = 1, \ldots, k$. Then, $d_k(n)$ is a simple sum of such values of ancestors of the lowest leaf.

Let $ch(v)$ be the set of children of v. Let $p^i(v)$ be v when i equals 0 and the parent of $p^{i-1}(v)$ when i is greater than or equal to 1. Let $desc_i(v)$ be the number of descendants of v of distance at most i. Thus, $desc_0(v) = 1$, counting the node itself. For convenience, define $desc_{-1}(v) = 0$. Let $depth(v)$ denote the distance from v to the root r, that is, the number of edges on the path.

ALGORITHM CHROMATIC POLYNOMIAL

```
ChromPoly(T)
{ Input: Tree T }
{ Output: d_k(1), ..., d_k(n) }

arbitrarily root T at some node r;
order the nodes of T in a BFS order v_1, ..., v_n;
for each v ∈ V do
    desc_0(v) ← 1;
    desc_i(v) ← 1 + ∑_{u ∈ ch(v)} desc_{i-1}(u), i = 1, ..., k;
for j = n downto 1 do
    d(j) ← 0;
    for i = 1 to min(k, depth(v)) do
        d(j) ← d(j) + (desc_{k-i}(p^i(v_j)) − desc_{k-i-1}(p^{i-1}(v_j)));
    for i = 1 to min(k, depth(v)) do
        decrease desc_{k-i}(p^i(v_j)) by one;
for i ← 1 to n do
    output d(i);
```

The time complexity of ALGORITHM CHROMATIC POLYNOMIAL is clearly $O(nk)$.

From the BFS-order of ALGORITHM CHROMATIC POLYNOMIAL we can, in fact, deduce more as shown in the next proposition.

Proposition 5.6 ALGORITHM CHROMATIC POLYNOMIAL *correctly computes the sizes d_i of the strongly simplicial neighborhoods of T^k.*

Proof. The number of nodes within distance k from v that have a common ancestor at $p^i(v)$ equals the number of descendants of $p^i(v)$. Thus, the number of nodes within distance k from v that have $p^i(v)$ as a common ancestor but not $p^{i-1}(v)$ equals $desc_{k-i}(p^i(v)) − desc_{k-i-1}(p^{i-1}(v))$. This is counted at each inner iteration in the algorithm. The resulting sum then

23

correctly counts once all nodes within distance k from v. In the second inner loop, we decrement all counters that involve the simplicial node being removed. $\square$

REMARK: For fixed n, the number of vertices of T, we need at least $O(n)$ calculations to evaluate $\chi_{T^k}(t)$ for any given t. Hence, the best algorithm possible must be $O(f(k)n)$, where f is some function.

Acknowledgments

References

[1]   R. Balakrishnan and P. Paulraja. Powers of Chordal Graphs. *Australian Journal of Mathematics*, Series A, **35**:211-217, (1983).

[2] C. Berge. Balanced Matrices. *Mathematical Programming*, **2**:19–31, (1972).

[3] J. Ch. Boland and C. G. Lekkerkerker. Representation of Finite Graphs by a Set of Intervals on the Real Line. *Fundamentals of Mathematics*, **51**:45–64, (1962/1963).

[4] D. G. Corneil and P. E. Kearney. Tree Powers. *Journal of Algorithms*, **29**:111–131, (1998).

[5] E. Dahlhaus and P. Duchet. On Strongly Chordal Graphs. *Ars Combinatoria*, **24 B**:23–30, (1987).

[6] P. Damaschke. Distances in Cocomparability Graphs and Their Powers. *Discrete Applied Mathematics*, **35**:67–72, (1992).

[7]   P. Duchet. Propriété de Helly et Problémes de Représentation. *Colloque C.N.R.S.*, (Orsay 1976), CNRS, Paris, **260**:117–118, (1978).

[8] M. Farber. Characterization of Strongly Chordal Graphs. *Discrete Mathematics*, **43**:173–189, (1983).

[9] M. Farber. Domination, Independent Domination, and Duality in Strongly Chordal Graphs. *Discrete Applied Mathematics*, **7**:115–130, (1984).

[10] U. Feige and J. Kilian. Zero Knowledge and the Chromatic Number. *Journal of Computer and System Sciences*, **57**:187–199, (1998).

[11] F. Gavril. The Intersection Graphs of Subtrees in Trees are Exactly the Chordal Graphs. *Journal of Combinatorial Theory B*, **16**:47-56, (1974).

[12] M. M. Halldórsson, J. Kratochvíl, and J. A. Telle. Independent Sets with Domination Constraints. *Discrete Applied Mathematics*, **99**:39–54, (2000).

[13] N. Kalyana, R. Prasad, and P. S. Kumar. On Generating Strong Elimination Orderings of Strongly Chordal Graphs. *Proceedings of the 18th Foundations of Software Technology and Theoretical Computer Science Conference.* Springer-Verlag, Berlin, Germany, pages 221–232, (1998).

[14] R. Laskar and D. Shier. On Chordal Graphs. *Congressus Numerantium*, **29**:579–588, (1980).

[15] R. Laskar and D. Shier. On Powers and Centers of Chordal Graphs. *Discrete Applied Mathematics*, **6**:139–147, (1983).

[16] Y.-L. Lin and S. Skiena. Algorithms for Square Roots of Graphs. *SIAM Journal of Discrete Mathematics*, **8**(1):99–118, (1995).

[17] S. L. Peng and M. S. Chang. A Simple Linear Time Algorithm for the Domatic Partition Problem on Strongly Chordal Graphs. *Information Processing Letters*, **43**:297–300, (1992).

[18] A. Raychaudhuri. On Powers of Interval and Unit Interval Graphs. *Congressus Numerantium*, **59**:235–242, (1987).

[19] A. Raychaudhuri. On Powers of Strongly Chordal and Circular Graphs. *Ars Combinatoria*, **34**:147–160, (1992).

[20] D. B. West. Introduction to Graph Theory. *Prentice-Hall Inc.*, Upper Saddle River, New Jersy, (1996).