Minimizing Interference of a Wireless Ad-Hoc Network in a Plane

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Abstract

We consider interference minimization in wireless ad-hoc networks. This is formulated as assigning a suitable transmission radius to each of the given points in the plane, so as to minimize the maximum number of transmission ranges overlapping any point. Using ideas from computational geometry and ϵ-net theory, we attain an $O(\sqrt{\Delta})$ bound for the maximum interference where Δ is the interference of a uniform-radius ad-hoc network. This generalizes a result given in [16] for the special case of highway model (i.e., one-dimensional problem) to the two-dimensional case. We show how a distributed algorithm can achieve a slightly weaker bound. We also give a method based on quad-tree decomposition and bucketing that has another provable interference bound in terms of the ratio of the minimum distance to the radius of a uniform-radius ad-hoc network.

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1 Introduction

Wireless ad-hoc and sensor networks are emerging areas of active research. Since energy is the limiting factor for the operability and lifetime of these networks, various mechanisms have been developed to conserve energy. These are collectively called topology control.

In a common formulation, each device represents a point (or *node*) in the Euclidean plane, and each node has a disk of a given transmission radius. Two nodes can communicate with each other if they are located within each other's disks; symmetric communication is considered essential to reduce protocol complexity. We assume that the transmission radius of a node is a controllable parameter and a monotone function of the electric power given to the node. Topology control involves assigning a suitable transmission radius to each node to form a connected network while minimizing some non-decreasing objective function of the radii.

A primary issue in wireless communication is *interference*, where communication between two parties is affected by transmissions from a third party. High interference increases the probability of packet collisions and therefore packet retransmission, which can significantly affect the effectiveness of the system and the energy use. Therefore, it is desirable to keep a low interference at every node.

Traditionally, interference has been implicitly minimized by reducing the density or the node degrees in the communication network. By keeping the transmission radii small, we then not only reduce the power consumption but also the density, and intuitively also the interference. Burkhart et al. [4], however, showed that low interference is not implied by sparseness. Also, that networks constructed from nearest-neighbor connections can fail dismally to bound the interference. On the other hand, they gave experimental results that indicate that graph spanners do help reduce interference in practice. Their work prompted the explicit study of interference minimization.

Several possible models of interference have been studied. The model of [4] measures the number of nodes affected by the communication of a single communication link. This was also studied by Moaveni-Nejad and Li [13], who further introduced the measure of the number of receiving nodes affected by the communication from a single sender. Both problems were further studied by Benkert et al. [2]. In both cases, the problems can be solved optimally by using MST computation.

Von Rickenbach et al. [16] argued that a sender-centric model of interference was misguided, since the interference was actually felt by the receiver. Further, that it was overly sensitive to the addition of single nodes. Instead, the formulated the problem studied here of minimizing the maximum interference received at a node. They gave algorithms for the special case where all the points are located on a line, called the *highway model.* Their algorithm constructs a network with an $O(\sqrt{\Delta})$ interference, where Δ is the interference of a uniform radius network, and they shoed that there exists an instance that requires $\Omega(\sqrt{n})$ interference. They also showed that the better of a naive network and the above $O(\sqrt{\Delta})$ interference network attains a $O(\Delta^{1/4})$ approximation ratio. This left open both the question of the hardness of the problem and of its approximability in more general scenarios.

Further recent work has been done on interference minimization. The related problem of bounding the *average* interference received at a node was considered by Moscibroda and Wattenhofer [14], who gave a nearly tight logarithmic approximation algorithms. In the same conference proceedings as the original version of the current paper [5], Bilò and Proietti [3] analyzed the approximability these interference problems under general distance functions, and gave logarithmic lower bounds for all of them. Also, for all but our problem, they gave logarithmic approximation algorithms. Johansson and Carr-Motyčková [7] introduced interference metrics based on averages over the edges of communication paths between nodes in the network. They gave simulation results for these and the aforementioned metrics. An upper bound on the approximation for a different kind of a receiver-centric interference problem was given by Kuhn et al. [8], where the task is to select a subset of the nodes as backbone stations and the interference is only caused and measured by the backbone stations. There is no modeling of a connectivity requirement of this backbone. They formulated this for arbitrary distance functions as a *minimum membership set cover problem*, and gave a logarithmic approximation algorithm, based on randomized rounding of a linear programming solution.

Our results. We present in this paper the first results on the maximum intereference minimization at receiving nodes in the two-dimensional case. In particular, we show that we can construct a network with an $O(\sqrt{\Delta})$ interference for any point set in the plane, extending the theory of [16] to the planar case (and actually for any constant-dimensional space). The construction is simple, except that it needs an ϵ-net as its part. If we use the theoretically optimal ϵ-net as a component, we obtain the $O(\sqrt{\Delta})$ bound. We can also use a random sample as the ϵ-net to obtain a simple and distributed construction, for a slightly weaker interference bound of $O(\sqrt{\Delta \log \Delta})$. Moreover, we give a network with an $O(\log(R_{min}/d))$ interference, where d is the minimum distance between points and R_{min} is the minimum radius of a uniform-radius network to attain connectivity. Our results rely on computational geometric tools such as local neighbor graphs, ϵ-nets, and quad-tree decompositions.

2 Preliminaries

We are given a set $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of points in a plane with Euclidean distance function d. For each $\mathbf{v}_i$, we are to assign a positive real number $r(\mathbf{v}_i)$ called the *transmission radius*. This can be viewed as a *radius assignment* function $r : V \to \mathbb{R}^+$, giving the set $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ of disks, where D_i has radius $r(\mathbf{v}_i)$ and center at $\mathbf{v}_i$.

We define a *wireless network* on V based on r as the graph $G(\mathcal{D}) = (V, E)$, with an undirected edge $(\mathbf{v}_i, \mathbf{v}_j)$ if and only if $\mathbf{v}_i \in D_j$ and $\mathbf{v}_j \in D_i$. In other words, $\mathbf{v}_i$ and $\mathbf{v}_j$ can directly communicate since they are within the transmission radius of each other. We say that the network $G(\mathcal{D})$ is *feasible* iff it is connected.

The interference of $\mathcal{D}$ at a point $\mathbf{p}$ is the number of disks in $\mathcal{D}$ covering $\mathbf{p}$. That is,

$$I(\mathcal{D}, \mathbf{p}) = |\{i : \mathbf{p} \in D_i\}|.$$

The *interference* of a network $G(\mathcal{D})$ is [3]

$$\max\{I(\mathcal{D}, \mathbf{p}) | \mathbf{p} \in \mathbb{R}^2\}.$$

The *interference minimization problem* is to find a radius assignment that yields a feasible network with minimum interference.

One natural approach is to increase all radii uniformly until the graph becomes connected. Let R_{min} be the infimum of the radius such that the network becomes connected, and refer to the network with all radii set to R_{min} as the *uniform-radius* network. Let Δ denote the interference of the uniform-radius network.

Although the problem is clearly an NP-optimization problem, it appears very difficult to find the optimal wireless network. Indeed, even the special case where all points V are located on a line (highway model) is considered difficult (although NP-hardness result is not known). Thus, we seek a practical solution with some theoretical quality guarantee, either as an upper bound of the interference or as an approximation ratio.

[3] We can also consider the version where we only consider interference at points of V, not all points in the plane. The results of this paper carry immediately over to that model.

2.1 Review for the Highway Model

We briefly review some results for the highway model given by von Rickenbach et al. [16]. Suppose that points of V are located on the x-axis in the sorted order with respect to their x-values.

A naive method is to set $r(i) = \max(d(\mathbf{v}_i, \mathbf{v}_{i-1}), d(\mathbf{v}_i, \mathbf{v}_{i+1}))$ for $i = 1, 2, \ldots, n$, where we set $\mathbf{v}_0 = \mathbf{v}_1$ and $\mathbf{v}_{n+1} = \mathbf{v}_n$. It is easy to observe that $G(\mathcal{D})$ associated with this radius function is feasible: the network is called the *linear network*. The linear network has interference at most Δ and works well on typical practical instances, for example, on a randomly distributed point set. Unfortunately, there is an instance for which the linear network poorly performs. In the *exponential chain* forming this instance, the points satisfies that $d(\mathbf{v}_i, \mathbf{v}_{i+1}) = 2^i$ for $i = 1, 2, \ldots n-1$, and it is easy to see that the interference of the point v_1 is $n-1$ in the linear network.

We can use a *hub-connected* network to reduce the worst-case interference. The general idea is as follows. We find a subset $W \subset V$ of points called *hubs* and first construct a linear network of the hubs. Then, for each $\mathbf{v} \in V \setminus W$, we set

$$r(\mathbf{v}) = \min_{\mathbf{w} \in W} d(\mathbf{v}, \mathbf{w});$$

namely, each non-hub connects to its nearest hub. If we select every $\sqrt{n}$-th point in V as a hub, we have a set W of cardinality $\sqrt{n}$, and can show that $I(G(\mathcal{D})) = O(\sqrt{n})$ for this network. It has been shown that the minimum interference is $\Omega(\sqrt{n})$ for the exponential chain, thus the hub-connected network is worst-case optimal. However, for each given instance, we can often design a network with a better interference. Indeed, there is a simple extension of this construction with $I(G(\mathcal{D})) = \sqrt{\Delta}$.

3 Two-Dimensional Ad-Hoc Network with Low Interference

3.1 Two-Dimensional Analogue of the Linear Network

Although the linear network performs poorly in the worst case for the highway model, it is a basic structure that can also be constructed in a distributed fashion. That is, each point can connect to its right and left neighbors without the need for global information.

The first task is to extend this notion to the two-dimensional case, where there are no clear notions of left and right neighbors. If we sort the points with respect to x-coordinate, and each point connects to the nearest neighbor with respect to the x-coordinate, we can obtain a feasible network. However, this ignores the y-coordinate and usually gives a bad network. Instead, we would like to use the Euclidean distance to measure the proximity of points.

Indeed, a network in which each node establishes a (two-way) connection with its nearest neighbor is called a *nearest-neighbor forest*. The nearest-neighbor forest need not be connected, however, and we want give a connected network based on it. The minimum spanning tree MST(S) might be a direct two-dimensional analogue of the linear network. The wireless version is WMST(S) in which each node $\mathbf{p}_i$ has the radius $\max_{\mathbf{q}:(\mathbf{p}_i,q)\in \mathrm{MST}(S)} d(\mathbf{p}_i, q)$. The minimum spanning tree has been widely considered as a structure of ad-hoc wireless networks, and is reported to work well for practical inputs [4].

Constructing a minimum spanning tree explicitly requires some global information. Hence, we want to consider another graph with a more local nature as a two-dimensional extension of the linear network. We briefly explain the *local neighborhood graph* (LNG) [18], which inspires the construction of our hub-structure network given later.

For each point $\mathbf{p} \in \mathbb{R}^2$, we divide the plane into six cones $R_1(\mathbf{p})$, $R_2(\mathbf{p})$, $\ldots, R_6(\mathbf{p})$, where $R_k(\mathbf{p})$ is the region such that the argument angle about $\mathbf{p}$ is in the range $[\frac{(k-1)\pi}{3}, \frac{k\pi}{3})$. Let $nb_k(\mathbf{p}, V)$ be the nearest point to $\mathbf{p}$ in $V \cap R_k(\mathbf{p})$. See Fig. 1. The local neighbor graph LNG(V) is the graph connecting each $\mathbf{v} \in V$ to its six local neighbors.

Lemma 3.1 *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* $R_k(\mathbf{p})$ *and* $d(\mathbf{p}, \mathbf{u}) \leq d(\mathbf{p}, \mathbf{v})$. *Then,* $d(\mathbf{u}, \mathbf{v}) < d(\mathbf{p}, \mathbf{v})$.

Proof: Straightforward from the fact that the diameter (distance between the farthest pair of points) of a fan with the angle $\pi/3$ equals the radius of the circle. □

The above lemma is known to lead to the fact that LNG(V) contains MST(V) and is therefore connected [18].

Let $N_{out}(\mathbf{v}) = \{nb_k(\mathbf{v}, V) | 1 \leq k \leq 6\}$ denote the nearest neighbors of $\mathbf{v}$ in each of its cones. Also, let $N_{in}(\mathbf{v}) = \{\mathbf{w} \in V | \mathbf{v} \in N_{out}(\mathbf{w})\}$ be the vertices that have $\mathbf{v}$ has their nearest neighbor. If we set $r_i = \max\{d(\mathbf{v}_i, \mathbf{q}) | \mathbf{q} \in N_{out}(\mathbf{v}_i) \cup N_{in}(\mathbf{v}_i)\}$, for each $i = 1, 2, \ldots n$, we have a network WLNG(V) that contains LNG(V) as a subgraph. Note that we need $N_{in}(\mathbf{v})$ since we need to answer connection requests from $\mathbf{w} \in N_{in}(\mathbf{v})$ to establish the bidirectional connection.

We remark that WLNG(V) can be constructed locally: Each node increases its radius (up to a given limit) and sends a message until it receives acknowledgement from the local neighbor in each of its six cones, and sends a connection

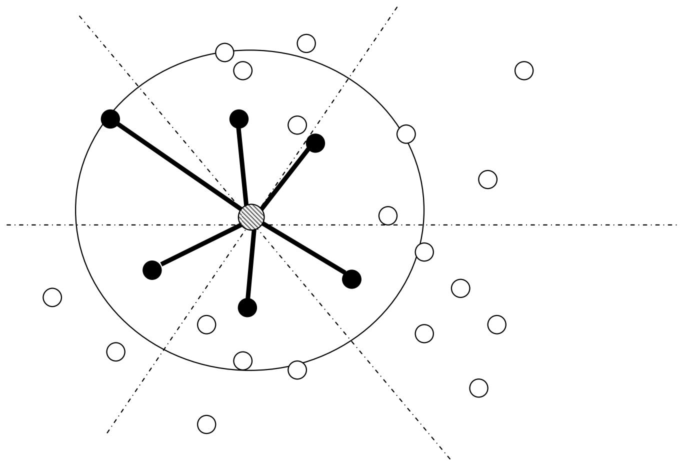

Fig. 1. Local neighbors of a point and a disk connecting them

request to each local neighbor. Then, each node that receives a connection request increases the radius until it can reach the sender. We remark that this method has the weakness that we need to set a limit radius, since if there is an empty cone, we have to detect it and ignore it to avoid increasing the radius to infinity. This will be resolved using a localization method given in the subsequent sections, where the limit of the radius is set to R_{min}, the radius of the uniform-radius network.

3.2 Hub-Connected Network with $O(\sqrt{n})$ Interference

It is known that there is an instance for which any network containing the nearest-neighbor forest has an $\Omega(n)$ interference even though there exists a constant interference network for the instance [16]. Thus, if every node connects to its nearest neighbor, we can obtain neither a (nontrivial) absolute interference bound nor a good performance ratio.

In order to attain a better interference bound, we consider a hub-connected network, where we select a subset W of V as a set of hubs. We construct $\mathrm{WMST}(W)$ as the core of the network, and propagate the connection around the core such that each vertex $\mathbf{v} \in V \setminus W$ is connected to its nearest hub. Note that we may use any connected network on W (e.g., $\mathrm{WLNG}(W)$) as a core instead of $\mathrm{WMST}(W)$ in order to attain our main theoretical result; what matters only is the choice of W.

3.2.1 Hub Selection Using an ϵ-Net.

To define the set W of hubs we turn to the concept of ϵ-nets. Consider a family $\mathcal{R}$ of regions in the plane. Given a set V of n points, the pair $(V, \mathcal{R})$ is called a *range space*. For any given $\epsilon < 1$, an *ϵ-net* of the range space $(V, \mathcal{R})$ is a subset $S \subset V$ such that any region $R \in \mathcal{R}$ that contains at least ϵn points of V must contain at least one point of S. That is,

$$|V \cup R| \geq \epsilon n \quad \Longrightarrow \quad S \cup R \neq \emptyset$$

holds for every $R \in \mathcal{R}$. The value ϵ can be any positive real number less than 1 and may depend on the size of V. Intuitively, an ϵ-net is a uniformly distributed sample of V, but the uniformity is measured using the family $\mathcal{R}$ of regions. For instance, if $\mathcal{R}$ is the family of all halfplanes, the set of points on the boundary of the convex hull of V becomes an $1/n$-net, since any halfplane containing a point of V must also contain a point on the convex hull.

The following theory (which the reader need not be familiar with) has numerous applications in computational geometry [1] and learning theory: The *Vapnik-Chervonenkis*-dimension (VC dimension) of a range space is the largest size of a subset $A \in V$ such that all subsets of A are attained as an intersection of A and a region in $\mathcal{R}$. If the VC dimension is low (say, a constant), we can always obtain a small ϵ-net (see [11] for example).

Here, we consider a range space associated with a family of regular triangles. Here, a triangle means the closed region bounded by its three edges. Consider the regular triangle P_1 spanned by $(0,0)$, $(1,0)$, $(1/2, \sqrt{3}/2)$. Let P_2 be the reflected image of P_1 with respect to the x-axis. The family $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) is the set of all translated and scaled copies of P_1 (resp. P_2). Concretely, let $P_1(\mathbf{p}, s)$ be the triangle spanned by $\mathbf{p}$, $\mathbf{p} + (s, 0)$, $\mathbf{p} + (s/2, \sqrt{3}s/2)$ that is obtained by translating P_1 by a vector $\mathbf{p}$ and scaled by s (fixing $\mathbf{p}$ as its vertex). Then, $\mathcal{P}_1 = \{P_1(\mathbf{p}, s) : \mathbf{p} \in \mathbb{R}^2, s \in \mathbb{R}^+\}$, and $\mathcal{P}_2$ is the set of reflected triangles of those in $\mathcal{P}_1$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

First, we give a weaker bound on the size of an ϵ-net of $\mathcal{P}$. Although this will be slightly improved later, the following result is useful since we do not need any complicated algorithm to find the ϵ-net. In particular, this gives an easy local (fully distributed) algorithm. The following theorem is a fundamental theorem in learning theory and computational geometry [10,9].

Theorem 3.2 *Let $(V, \mathcal{R})$ be a range space with V finite and of finite VC-dimension d. Then, a random sample $S \subset V$ of size $C(d) r \log r$ is a $1/r$-net for $(V, \mathcal{R})$ with probability whose complement to 1 is exponentially small in r. The constant $C(d)$ depends only on d.*

It is known that the VC dimension of the set of all triangles in the plane is finite [10]. Therefore, the VC dimension of $\mathcal{P}$ is also finite, since the VC dimension of a subfamily is at most that of the original family. Thus, we obtain the following corollary:

Corollary 3.3 *A random sample of size $c\epsilon^{-1}\log\epsilon^{-1}$ becomes an ϵ-net for $\mathcal{P}$ with high probability if c is a sufficient large constant.*

A family $\mathcal{R}$ of regions is said to be a family of *pseudo-disks* if for any three non-collinear points in the plane, there exists a unique $R \in \mathcal{R}$ such that those three points are on the boundary of R. The following better bound is known for a family of pseudo-disks.

Theorem 3.4 *[12] For any point set V, there is an ϵ-net of size $O(1/\epsilon)$ for a family of pseudo-disks.*

Consider the family $\mathcal{P}_k$ for $k = 1, 2$, say, $k = 1$. We say that a point set satisfies the non-degeneracy condition if no two points lie on a horizontal line, a vertical line, or a line with argument angle $\pi/3$. It is easy to see that for any three points satisfying the non-degeneracy condition, there exists at most one $P \in \mathcal{P}_1$ such that the triple of points are on the boundary of P. Thus, $\mathcal{P}_1$ has a property that is very similar to pseudo-disks, but there may be noncollinear triplets that are not contained on the boundary of any $P \in \mathcal{P}$. Nevertheless, we have the following theorem that improves slightly on Corollary 3.3:

Theorem 3.5 *There exists a polynomial-time computable ϵ-net of size $O(1/\epsilon)$ for $(V, \mathcal{P})$.*

This theorem is of independent interest within computational geometry. In comparison with the random sampling method of Corollary 3.3, the construction of the ϵ-nets is quite complicated and also difficult to compute in a distributed fashion. Thus, the random sampling method is preferable in practice, and the rest of the paper is complete without Theorem 3.5 if we increase the interference by a $\sqrt{\log n}$ factor (see Theorem 3.7 and Theorem 3.8). We therefore give the construction of an ϵ-net establishing Theorem 3.5 in a later section for readers interested in computational geometric theory.

3.2.2 The Hub-Connected Network.

The construction is as follows: We first compute an $\sqrt{n^{-1}}$-net W of V of size $O(\sqrt{n})$ using Theorem 3.5, by setting $\epsilon = \sqrt{n^{-1}}$. We then form any connected network on W (e.g., WMST(W)), and let $r_0(w)$ be the transmission radius of $w \in W$ in that network.

We call the elements of W *hubs*. For each non-hub $v \in V \setminus W$, we find its nearest hub, denoted by $hub(v)$, and set $r(v) = d(v, hub(v))$. For each hub

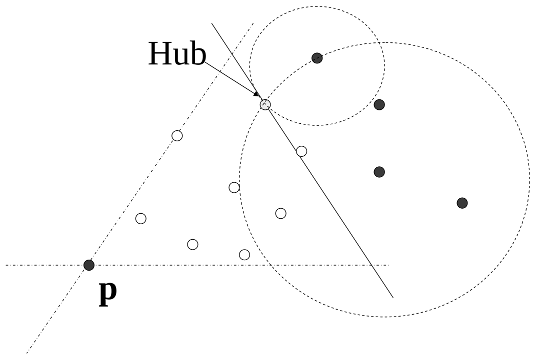

Fig. 2. No disk around a point outside the region $P(\mathbf{w})$ can reach $\mathbf{p}$

$w \in W$, define the set $N(w) = \{v \in V \setminus W | hub(v) = w\}$ of nodes using that hub, and set $r(w) = \max\{r_0(w), \max_{v \in N(w)} d(v, w)\}$. This determines r for each element of V, giving a wireless network GHUB(V).

Lemma 3.6 GHUB(V) *is connected.*

Proof: Since WMST(W) is connected, the subgraph of GHUB(V) induced by W is connected. Since other nodes are all connected to nodes in W, GHUB(V) is connected. □

Theorem 3.7 *The interference of* GHUB(V) *is* $O(\sqrt{n})$.

Proof: Let c be a suitable constant such that $|W| < c\sqrt{n}$. We claim that any point $\mathbf{p} \in \mathbb{R}^2$ is covered by at most $(c + 6)\sqrt{n}$ disks, or, more precisely, by $6\sqrt{n}$ disks excluding those around elements of W.

Consider the cusp $R_1(\mathbf{p})$ whose argument angle interval is $[0, \pi/3)$. By symmetry, it suffices to show that at most $\sqrt{n}$ points in $R_1(\mathbf{p})$ can contain $\mathbf{p}$ in their disks. If there is no hub in $R_1(\mathbf{p})$, then $R_1(\mathbf{p})$ cannot contain more than $\sqrt{n}$ points because W is a $\sqrt{n^{-1}}$-net, and we are done. Otherwise, we can assume there is at least one hub in $R_1(\mathbf{p})$ (see Fig. 2). Consider a hub $\mathbf{w} \neq \mathbf{p}$ in $R_1(\mathbf{p})$. We draw a line of argument angle $2\pi/3$ through $\mathbf{w}$ such that it makes a regular triangle $P(\mathbf{w}) \in \mathcal{P}_1$ together with the two boundary lines of $R_1(\mathbf{p})$. We select the hub $\mathbf{w}$ such that $P(\mathbf{w})$ is minimized. Then, $P(\mathbf{w})$ does not contain a hub in its interior, and hence $P(\mathbf{w})$ can contain at most $\sqrt{n}$ elements of V. Consider any point $\mathbf{x} \in V$ in $R_1(\mathbf{p}) \setminus P(\mathbf{w})$. Then, we can see that $d(\mathbf{x}, \mathbf{w}) < d(\mathbf{x}, \mathbf{p})$ analogously to Lemma 3.1. Since $r(\mathbf{x})$ is the distance to its nearest hub, $r(\mathbf{x}) \leq d(\mathbf{x}, \mathbf{w}) < d(\mathbf{x}, \mathbf{p})$. Thus, $\mathbf{p}$ is not in the disk of $\mathbf{x}$. We can deal with the other five cusps similarly. This completes the proof. □

We can use a random sample as the set of hubs to obtain a slightly weaker result.

Theorem 3.8 *If we use a random sample of size* $\sqrt{n \log n}$ *as the set* W *of hubs in the above construction, the interference of* GHUB(V) *is* $O(\sqrt{n \log n})$ *with high probability.*

Proof: From Corollary 3.3, the random sample is an $O(\sqrt{n^{-1} \log n})$-net with high probability. The rest of the analysis is analogous to Theorem 3.7. □

If we use a random sample for the hub set W as in Theorem 3.8 and use WLNG(W) instead of WMST(W) (recall that any connected network on W can be used), the corresponding version of GHUB(V) can be constructed in a distributed fashion: First, each point recognizes itself as a hub with probability $\sqrt{n^{-1} \log n}$ independently. This determines the set W of hubs. Next, each point of $V \setminus W$ independently finds its nearest hub by enlarging its radius until it receives a response from a hub node. Each hub enlarges its radius, if necessary, to the distance to the furthest node that requests a connection to it. In parallel, WLNG(W) is constructed, and its construction can be done locally as discussed before.

4 A Network with $O(\sqrt{\Delta})$ Interference

Let us consider the uniform-radius network G_0 in which each disk has the same radius R_{min}. Recall that Δ is the interference of G_0. Although Δ can become as large as $\Omega(n)$, it can in practice be much smaller than n, or even $\sqrt{n}$. We show a construction of a network where the interference is $O(\sqrt{\Delta})$.

We use a standard localization method by bucketing. By scaling, we can assume that $R_{min} = 1$ to eliminate one parameter. We partition the plane into unit square buckets by an orthogonal grid. For simplicity of argument, we assume that there are no points on boundaries of buckets; this assumption is easy to remove. We say that two buckets B and B' are adjacent if there exists $v \in B$ and $v' \in B'$ such that the edge (v, v') is in G_0.

Lemma 4.1 *(1) For each bucket, an adjacent bucket must be one of its eight neighbors in the grid.*
(2) Each bucket contains $O(\Delta)$ *points.*

Proof: Statement 1 is obvious, since the distance from any point in B to any bucket other than the eight neighbors is more than 1. For statement 2, suppose that a bucket contains more than 4Δ points. We refine the buckets into four sub-buckets of size 0.5×0.5. One of the sub-bucket contains more than Δ points, and the center of the sub-bucket is covered by the unit disk about each point in its sub-bucket. This contradicts the assumption that the interference of G_0 is Δ. □

Our construction is as follows: First, in each bucket B, we give a network with interference $O(\sqrt{\Delta})$ using the construction given in the previous subsection, and set the radius of each point accordingly. Note that none of the disks in the construction has a radius larger than $\sqrt{2}$. Second, for each adjacent pair B, B' of buckets, select exactly one edge $(v, v') \in G_0$ connecting them. We call v and v' *connectors*. We enlarge the radius of each connector to 1 (if its current radius is less than 1) .

Now, we have defined all the radii, and accordingly we have a network LHUB(V).

Theorem 4.2 *The network* LHUB(V) *is connected, and its interference is* $O(\sqrt{\Delta})$.

Proof: The network is connected within each bucket, and the connection between buckets is the same as in G_0. Thus, it is connected. For each point p, it is interfered by points of at most 21 buckets (the neighbor buckets of Manhattan distance at most 2), since the radius of the largest disk is at most $\sqrt{2}$. Each bucket contributes only $O(\sqrt{\Delta})$, excluding connectors. Also, there are only a constant number of connectors in these buckets. Thus, we have the theorem. □

We remark that we obtain a $O(\sqrt{\Delta \log \Delta})$ interference if we use the construction given in Theorem 3.8 for the network in each bucket.

5 A Hierarchical Construction

The GHUB network has two layers: hubs and non-hubs. The LHUB network has three layers: connectors, hubs in buckets, and others. One might think a better structure could be obtained if we increase the number of layers. The lower bound of $\Omega(\sqrt{\Delta})$ for the one-dimensional model shows however that we have a tight bound as a function of n or Δ. Still, this can be advantageous in practice, as we can see if we measure interference using a different parameter.

Let d be the minimum distance between two points in V. Below, we give a network whose interference is $O(\log(R_{min}/d))$-approximate, where R_{min} is the radius to give the uniform-radius network. As before, we scale the problem such that $R_{min} = 1$.

The same localization method as in Section 4 works here, so we may assume that all points are located in a unit square. Our approach is based on quad-tree decomposition. We adopt the convention that each square in the quad-tree decomposition includes its lower edge and its right edge, together with its lower two corner vertices.

We repeat the following process starting from $k = 0$, where $U(S) = V$ if $k = 0$:

Quad-tree decomposition process: Given a square S of size $2^{-k} \times 2^{-k}$ and a set $U(S) \subset V \cap S$, do the following.

(1) If $U(S) = \emptyset$, terminate the process.
(2) Otherwise, select a representative point $\mathbf{p}(S) \in V(S)$ arbitrarily, and remove $\mathbf{p}(S)$ from $U(S)$.
(3) Partition S into four quadrants of size $2^{-(k+1)} \times 2^{-(k+1)}$. The point set $U(S)$ is partitioned accordingly. The at most four non-empty quadrants obtained are called *children* of S.
(4) Apply the process iteratively to each child.

We call S' the parent of S if S is one of the children of S', and denote $S' = parent(S)$. We also say that $\mathbf{p}(S)$ is a child (resp. parent) of $\mathbf{p}(S')$ if S is a child (resp. parent) of S'. For the representative point $\mathbf{p}(S)$ of S, we set $r(\mathbf{p}(S)) = \max\{diag(S), d(\mathbf{p}(S), \mathbf{p}(parent(S))\}$, where $diag(S)$ is the length of the diagonal of the square S. Thus, we have assigned a radius to each point of V, and have a network QUAD(V).

Theorem 5.1 QUAD(V) *is connected, and its interference is* $O(\log d^{-1})$*, where d is the minimum distance between points of V.*

Proof: Since $r(\mathbf{p}(S)) \geq diag(S)$, the disk of $\mathbf{p}(S)$ contains all of its children. Also, $r(\mathbf{p}(S)) \geq d(\mathbf{p}(S), \mathbf{p}(parent(S)))$ means that the disk also contains its parent. Thus, the points are connected via the tree structure of the parent-child relation.

Now, let us analyze the interference at a point $\mathbf{p}$. There are at most $O(\log d^{-1})$ different sizes of squares in the quad tree decompositions, since the diagonal length of the parent square of a smallest square must be at least d (otherwise, it can contain only one point). Consider a bucket size 2^{-k}, and analyze how many representative points of such buckets can interfere with $\mathbf{p}$. The radius $r(\mathbf{p}(S))$ of a representative point of a square S of this size is at most $2^{-k+1}\sqrt{2}$, since the distance from the representative point to any point in the parent square is at most $diag(parent(S)) = 2^{-k+1}\sqrt{2}$. Thus, $\mathbf{p}(S)$ can interfere with $\mathbf{p}$ only if S intersects with the circle of radius $2^{-k+1}\sqrt{2}$ about $\mathbf{p}$. It is easy to see that there are only a constant number of such squares of this size. Thus, the interference at $\mathbf{p}$ is $O(\log d^{-1})$. $\square$

In a practical implementation, we should apply a routine to shrink each disk as much as possible while keeping the connection to its parent and children.

6 Construction of a Small-Size ϵ-Net

Here, we give a constructive proof of Theorem 3.5. It suffices to show the following:

Theorem 6.1 *There exists a polynomial-time computable ϵ-net of $(V, \mathcal{P}_\infty)$.*

Although the above theorem can be generalized for a family of all translated/scaled copies of any given convex region, we focus here on $\mathcal{P}_1$ (i.e., the region of translated and scaled copies of a given regular triangle) to avoid unnecessary abstraction. We remark that it is not difficult to observe that the construction gives a polynomial time algorithm using the fact that a *generalized Voronoi diagram* can be constructed in $O(n \log n)$ time [17]. However, we only show the construction algorithm and its correctness, and omit the time complexity analysis.

We modify the argument of [12] for a range space of pseudodisks so that it works for our range space. The modification itself is not a major one, but we give the whole argument in order to make the paper self-contained. We remark that the published conference version of [12] has an error in its proof, and a corrected proof is in an unpublished manuscript [4].

For simplicity, we assume the non-degeneracy condition that no two points of V lie on a horizontal line, a vertical line, or a line with argument angle $\pi/3$. We call a member of $\mathcal{P}_1$ a *range*, since we will use use the term "triangle" later for general triangles. For a range $P \in \mathcal{P}_1$, we define $Int(P)$ to be its interior. The boundary of P is $\partial(P) = P \setminus Int(P)$.

The following lemma holds in a more general setting where P is a convex body and P' is its scaled and translated copy. It is an easy exercise to prove it for our ranges (i.e., isothetic regular triangles) by a case study.

Lemma 6.2 *For any pair P and P' of ranges, $P \setminus P'$ is connected, and $\partial(P) \cap \partial(P')$ has at most two connected components. Moreover, under the non-degeneracy condition, given any set A of three points of V, there is at most one range containing A on its boundary.*

Given a point set S, we call a range P an *empty range* (with respect to S) if it contains no point of S in its interior. A pair of points $(\mathbf{p}, \mathbf{p}')$ of S is called

[4] This information, together with the address of the web page containing it, was given to the authors by J. Matoušek.

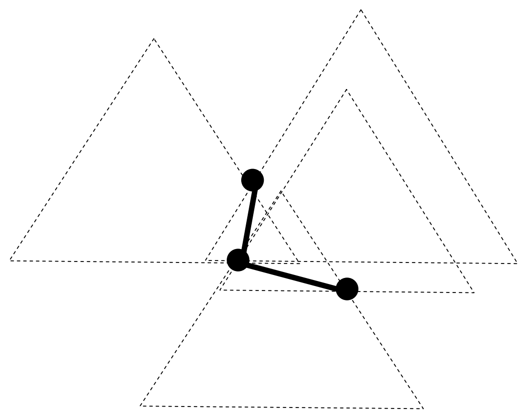

Fig. 3. A set of three points not contained on the boundary of any range.

a *Delauney pair* if there exists an empty range P containing $\mathbf{p}$ and $\mathbf{p}'$ on its boundary. A Delauney pair is called *extremal* if, for any number $N > 0$, there is an empty range P containing the pair on its boundary such that the area of P is larger than N.

Let $DT(S)$ be the graph consisting of a point set S (as the vertex set) and the set of all Delauney pairs (as the edge set). We draw each edge as the straight line segment between vertices.

Lemma 6.3 *Edges of $DT(S)$ intersect only at their endpoints.*

Proof: Let e and f be edges intersecting at an interior point. Let P and P' be empty ranges containing $e = (\mathbf{p}, \mathbf{p}')$ and $f = (\mathbf{q}, \mathbf{q}')$, respectively. Because S is nondegenerate, we can shrink P (resp. P') if necessary such that they contain no point in $S \setminus \{\mathbf{p}, \mathbf{p}'\}$ (resp. $S \setminus \{\mathbf{q}, \mathbf{q}'\}$). By the definition of empty ranges, $\mathbf{q}$ and $\mathbf{q}'$ (resp. $\mathbf{p}$ and $\mathbf{p}'$) are outside the interior of P (resp. P'). Let the edge e intersect $\partial(P \cap P')$ at points v_1 and v_2 and f intersect $\partial(P \cap P')$ at w_1 and w_2. If e and f intersect in the interior, these four points appear in a clockwise alternating order along $\partial(P \cap P')$, e.g. as v_1, w_2, v_2, w_2, since $P \cap P'$ is convex. Thus, $(P \cup P') \setminus Int(P \cap P')$ has four connected components. However, Lemma 6.2 implies that $(P \cup P') \setminus Int(P \cap P')$ has at most two connected components. We have a contradiction. □

Thus, $DT(S)$ gives a planar graph drawing. Indeed, it is the dual of the *generalized Voronoi diagram* [17]. We would like to claim that $DT(S)$ is a triangulation of S. This is known to hold for *pseudo disks* (assuming a suitable non-degeneracy condition) [12]. Unfortunately, $\mathcal{P}_1$ does not satisfy the condition of pseudo disks, and $DT(S)$ is not always a triangulation. Indeed, the set S of three black points in Fig. 3 does not have a range containing it on the boundary, and $DT(S)$ has only two edges, thus is not a triangulation.

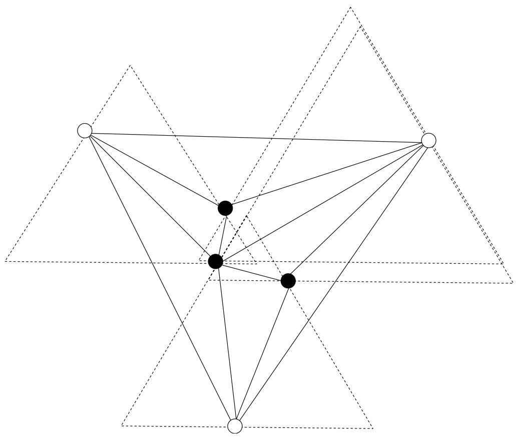

Fig. 4. Triangulation is obtained by adding the point set X.

We slightly modify the sets V and S to resolve the above problem. We add a set X of three "extra" points $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ to V. Let ℓ be a horizontal line that contains V in its lower halfplane. The points $\mathbf{q}_1$ and $\mathbf{q}_2$ are on the line ℓ, the point $\mathbf{q}_3$ lies below ℓ, and the three form vertices of a regular triangle (actually, one in $\mathcal{P}_2$). We take these three points sufficiently far from V so that X satisfies the following conditions:

(1) The triangle spanned by X contains all the points of V.
(2) For any range P, we have a range $P' \subseteq P$ such that $P' \cap V = P \cap V$, and $P' \cap X = \emptyset$.
(3) For any pair of points in X, there is a range P containing them on the boundary and containing no other point of V.
(4) For any extremal pair $(\mathbf{p}, \mathbf{p}')$ of a subset S of V, we have a range P with the largest size such that $Int(P) \cap (S \cup X) = \emptyset$ and $\{\mathbf{p}, \mathbf{p}'\} \in \partial P$. Note that a point of X must lie on the boundary of P, and intuitively, the point prevents $(\mathbf{p}, \mathbf{p}')$ to be an extremal pair in $S \cup X$.

We fix such an X. We write $\tilde{S}$ for $S \cup X$ for a subset S of V, and consider $DT(\tilde{S})$ instead of $DT(S)$. For the point set of Fig. 3, we obtain a triangulation by adding X (the three white points) as shown in Fig. 4.

Lemma 6.4 *$DT(\tilde{S})$ is a triangulation of the vertex set $\tilde{S}$ in the triangle spanned by X.*

Proof: Consider two points $\mathbf{p}, \mathbf{p}' \in S$ forming a Delauney pair. Given an empty range that has $\mathbf{p}$ and $\mathbf{p}'$ on its boundary, we can first shrink it so that one of $\mathbf{p}$ and $\mathbf{p}'$ comes to a vertex of the range. Thus, we can assume that $P = \triangle ABC$ is an empty range such that $\mathbf{p} = A$, and hence $\mathbf{p}'$ is on the edge BC because of nondegeneracy condition. We can grow P keeping the Delauney pair on the boundary. Indeed, there are two possibilities: one is the case where the triangle grows fixing B, and the other is the case where it

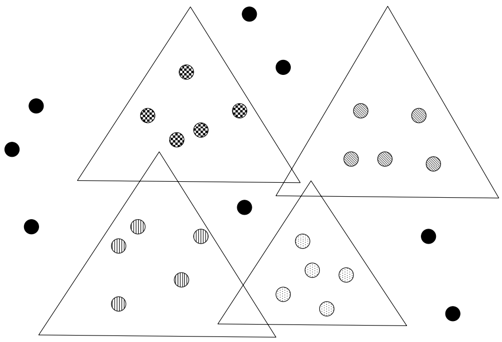

Fig. 5. Greedy procedure to find a maximal family of disjoint subsets of size δn.

grows fixing C. Since we have X in $\tilde{S}$, $(\mathbf{p}, \mathbf{p}')$ cannot be an extremal pair in $\tilde{S}$. Thus, we have two triangles in $DT(\tilde{S})$ that have $\mathbf{pp}'$ as their edges. Because of the non-crossing property of edges (Lemma 6.3), there is exactly one such triangle on each side of $\mathbf{pp}'$. Thus, $DT(\tilde{S})$ is connected and each interior face of the planar graph $DT(\tilde{S})$ must be a triangle. Thus, $DT(\tilde{S})$ is a triangulation of the triangle spanned by X. □

We call $DT(\tilde{S})$ the generalized Delauney triangulation of S. For each triangle in $DT(\tilde{S})$, the unique range P containing the three vertices of the triangle on its boundary is called its *Voronoi range*[5]. Note that a Voronoi range contains no point of $\tilde{S}$ in its interior.

Let $\delta = \epsilon/5$. We greedily find a maximal family of disjoint subsets $\{S_1, S_2, \ldots, S_k\}$ of V such that $|S_i| = \delta n$ and there exists a range P_i such that $P_i \cap V = S_i$. Fig. 5 shows such a family where $\delta n = 5$.

Let $S = \cup_{i=1}^{k} S_i$, and form $DT(\tilde{S})$. Any range P containing δn or more points of V must contain a point of S, since otherwise our family of subsets is not maximal. Thus, for each triangle in $DT(\tilde{S})$, there are at most δn points of V in its Voronoi range. Let D_i be the subgraph of $DT(\tilde{S})$ induced by S_i. Each triangle in D_i contains no point of V in its interior: It cannot contain a point of S because of the definition of $DT(\tilde{S})$, and it cannot contain a point of $V \setminus S$ since it is contained in the range P_i considered in the greedy process. The subgraph D_i is connected; otherwise, we can show that there is an empty range corresponding to an Delauney edge connecting two points in $S \setminus S_i$, and the intersection of the empty range and P_i violates Lemma 6.2.

Moreover, the union R of the triangles in D_i is simply connected. Here, a closed region in a plane is simply connected if it is connected and its complement is connected. R is in the convex hull of S_i. If the complement of R has more than one connected components, one connected component contains the exterior of

[5] This is an analogue of a Voronoi circle for an ordinary Voronoi diagram.

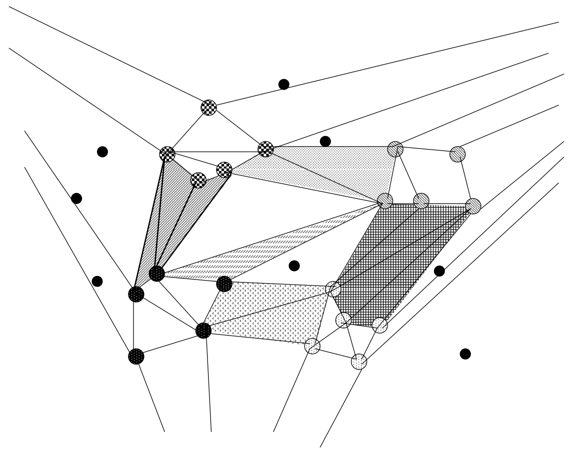

Fig. 6. Corridors in $DT(\tilde{S})$.

the convex hull, and the others (called *holes*) lies in the convex hull. A hole is a union of triangels of $DT(\tilde{S})$, and it must contain a triangle that has a point of $V \setminus S_i$ as a vertex. Thus, the convex hull contains a point of $V \setminus S_i$. This is a contradiction, since the range P_i defining S_i in the greedy procedure contains the convex hull of S_i, and all points in P_i must be in S_i.

We use $k+3$ colors to give a distinct color to each set S_i and to each of the three points of X. The points in $V \setminus S$ are colorless. We give corresponding colors to vertices of $\tilde{S}$. For two colors c_1, c_2, a triangle is called (c_1, c_2)-colored if its vertices use exactly those two colors.

For a fixed pair (c_1, c_2) of colors, we divide the set of (c_1, c_2)-colored triangles into maximal connected chains of triangles such that each pair of consecutive triangles share a bicolored edge. Such a maximal chain is called a *corridor*.

Lemma 6.5 *There are $O(k)$ corridors.*

Proof:

Since the union of triangles in D_i is simply connected, for each $i = 1, 2, \ldots, k$, we can contract each S_i of $DT(\tilde{S})$ into a point such that all bicolored edges in each corridor (say, corresponding colors of S_i and S_j) are replaced by an edge between S_i and S_j. This graph has k vertices, where each three-colored triangle remains a face in the new graph, while all other triangles are contracted. Thus, each face of the graph has three sides. Although this graph may have multiple edges as seen in Figure 7, it has at most $3f/2$ edges, where f is the number of faces. The number of edges is then $O(k)$, and the number of corridors is also $O(k)$. □

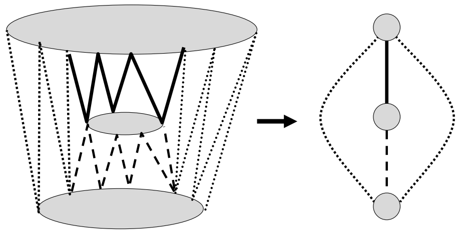

Fig. 7. The triangulation (where $k = 3$ and the triangulations of S_i ($i = 1, 2, 3$) are symbolized by ovals) given in the left picture is contracted to the graph given in the right.

The corridors are greedily refined into subcorridors containing at most δn points of V (indeed, they are colorless) in its triangles. Because the dual graph of each corridor is a tree of degree at most three, we can decompose it into $O(\frac{M}{\delta n})$ subcorridors if the corridor has M triangles. The vertex set of subcorridors consists of two monochromatic chains (possibly degenerated to points) in $D(S)$, and thus they have at most four endpoints. Let Z be the set of all endpoints of all subcorridors in $DT(\tilde{S})$.

Theorem 6.6 *Z is an ϵ-net of $V \cup X$, and its size is $O(1/\epsilon)$.*

Proof: The number of sub-corridors is $O(k+\frac{1}{\delta}) = O(1/\epsilon)$. Thus $|Z| = O(1/\epsilon)$. Consider any range P containing more than ϵn points of $V \cup X$. We assume that P contains no point of Z and derive contradiction. Without loss of generality, we assume that P contains at least one point in S_1 (colored red, striped in Fig. 8). $P \setminus P_1$ has at most one connected component because of Lemma 6.2, and let Y be the set of points in the component. If $Y = \emptyset$, P contains only red points, in which case it can have at most δn points; this is a contradiction. Thus, Y is nonempty.

Let C be the red monochromatic chain in the union of corridors. Since $P \setminus P_1$ is connected, there is a unique connected component C_1 of $C \cap P$ such that the other side has at least one point of (thus, all points of) Y. C_1 must be a subchain of a red chain C_{red} of a subcorridor, since P contains no point of Z. Let C_{blue} be the partner chain of the subcorridor, colored blue, the color of the set S_2. If C_{blue} intersects P (including the case that P contains no blue points but only intersect edges), there is no non-blue point below C_{blue} (i.e., different side from C_{red}), since otherwise $P \setminus P_2$ must have two connected components, contradicting Lemma 6.2. Let $e = (p_{red}, p_{blue})$ and $f = (q_{red}, q_{blue})$ be bicolored edges at the two ends of the subcorridor. The subcorridor is bounded by C_{red}, C_{blue}, e and f as shown in Fig. 9.

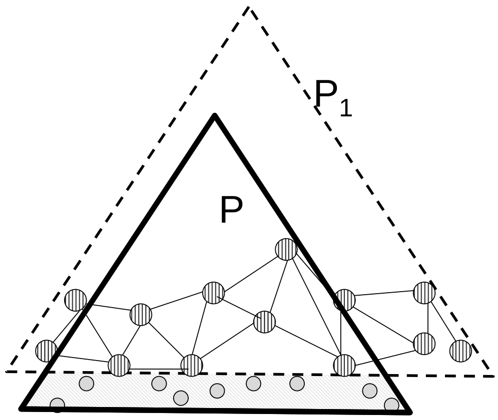

Fig. 8. Intersection of P and P_1, and the set Y (points in the shaded region).

If e intersects P, it must cut P into two pieces, since none of the endpoints of e are in P. Let R_e be the piece in the different side from the subcorridor. Similarly, we define R_f.

Let Q be the Voronoi range corresponding to the triangle containing e on the boundary (one of shaded triangles in Fig. 9) that is not in the corridor. If $R_e \setminus Q \neq \emptyset$, then $P \setminus Q$ has two connected components, one on each side of the edge e, contradicting Lemma 6.2. Thus, $R_e \in Q$, and R_e has at most δn points. R_f also has at most δn points.

The set of points in P consists of five parts. The part above or on C_{red} only has red points, thus at most δn points. The part below or on C_{blue} only has blue points, thus at most δn points. Each of R_e and R_f has at most δn (colorless) points. Finally, the subcorridor has at most δn (colorless) points. Thus, P has at most $5\delta n = \epsilon n$ points. □

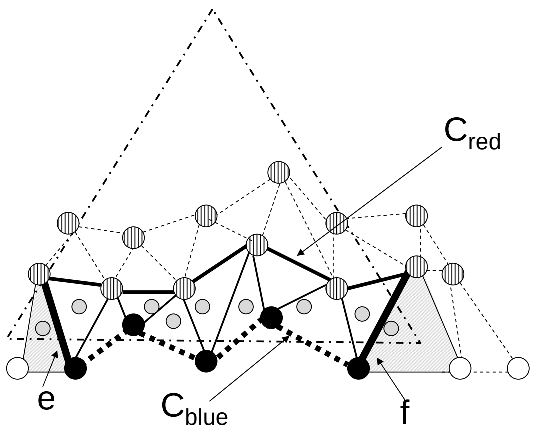

Fig. 9. A subcorridor intersecting P.

We finally show that $Z \setminus X$ is an ϵ'-net of V if $\epsilon n < \epsilon' n - 3$. Consequently, we have an ϵ-net of size $O(1/\epsilon)$ for the range space $(\mathcal{P}_1, V)$. Indeed, suppose we have a range P that contains $\epsilon' n$ points of V but no point in $Z \setminus X$. Then, it must contain one or more points of X. We can shrink P such that only the points of X go outside of it. This new range contains $\epsilon' n - 3$ points of V and no point of Z. This contradicts the fact that Z is an ϵ-net of $V \cup X$.

7 Concluding Remarks

The theory can easily be generalized to any constant dimensional space, except that we know only a $O(\epsilon^{-1} \log^{-1} \epsilon^{-1})$ bound for ϵ-nets of higher dimensional analogues of "range spaces of regular simplices". This leads to the construction of a network with an $O(\sqrt{n \log n})$ interference bound of a point set embedded in d-dimensional space, if d is a constant.

We can suggest several practical improvements to the method. For example, in the construction of QUAD(V), we can stop the partitioning if $|U(S)| = 1$, and otherwise partition $U(S)$ without selecting a representative point until there are at least two empty buckets. Also, we can mix the two methods: In each square S, we can replace the structure of the QUAD(S) network within S by LHUB(S), if it gives a better interference.

There are several open problems. One may observe that the exponential chain instance attains a $\Omega(\sqrt{\log(R_{min}/d)})$ lower bound in the highway model. We conjecture that this lower bound is tight, although we currently have only the $O(\log(R_{min}/d))$ upper bound given in this paper. Moreover, while for the highway model, the better of a linear network and a hub network attains a $O(\Delta^{1/4})$-approximation ratio to the optimal network, analogous result has not yet been obtained for the two-dimensional case.

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