Approximating the $L(h,k)$-labelling problem

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Abstract: The (h,k)-coloring problem, better known as the $L(h,k)$-labelling problem, is that of vertex coloring an undirected graph with non-negative integers so that adjacent vertices receive colors that differ by at least h and vertices of distance 2 receive colors that differ by at least k. We give tight bounds on approximations for this problem on general graphs as well as for bipartite, chordal, and split graphs.

Keywords: $L(h,k)$-labelling, distance-constrained coloring, frequency allocation, approximation algorithms

1 Introduction

The (h,k)-coloring problem, better known as the $L(h,k)$-labelling problem, is that of vertex coloring an undirected graph G with non-negative integers so that adjacent vertices receive colors that differ by at least h and vertices of distance 2 receive colors that differ by at least k. This problem was introduced by Griggs and Yeh [6] (in the case $h = 2$ and $k = 1$) to model a frequency assignment problem, where wireless transmitter/receivers must be assigned frequencies without causing interference.

A large body of research has developed through the years on (h,k)-coloring problems, particularly in relation to channel assignment in wireless networks. Most of that effort has been on two cases. The $(1,1)$-coloring problem is known as the *distance-2 coloring problem*, which again is closely related to coloring the square of a graph. The $(2,1)$-coloring problem is also known under the names λ-*coloring* and radio-coloring. A recent dynamically updated survey of Calamoneri [3] gives a thorough treatment of known results on exact solutions and bounds on (h,k)-coloring on different classes of graphs.

1.1 Our contributions

We are concerned here with the approximability of the (h,k)-coloring problem. In particular, we are interested in how the approximability varies with h and k. Thus, unlike many treatments where h and k are considered to be constants, we are primarily concerned with cases where they are growing functions of n, the number of vertices in the graph.

We give approximation algorithms and give approximation hardness reductions for (h,k)-coloring on both general graphs as well as some prominent classes of graphs. Given that the best performance ratio possible for these classes of graphs is a polynomial in n, we focus on the exponent for the polynomial, ignoring lower-order factors. We are able to derive the best possible exponent for the approximation of (h,k)-coloring for all values of h and k, both on general graphs as well as on bipartite, chordal, and split graphs. A scaling property shows that it is the ratio between h and k that matters, which allows us to assume without loss of generality that $k = 1$.

For general graphs, the optimal exponent is $1/2$ for $h \leq \sqrt{n}$, and grows after that linearly with h up to $h = n$. While the other three classes have the same approximability for $h = 1$, they show an interesting divergence as functions of h. For bipartite graphs the exponent stays also at $1/2$ for $h \leq \sqrt{n}$, but *decreases* linearly after that. For chordal and split graphs, the constant decreases uniformly with h, with the optimal performance ratio being about $\sqrt{n/h}$.

We illustrate the results graphically in Figure 1. We consider the performance ratio as functions of h, and draw on a logarithmic scale (with base n). The performance functions for bipartite, chordal/split, and general graphs are shown, with the lower and upper bounding matching in each case.

Our upper bounds are all based on a simple First-Fit algorithm, sometimes applied to a greedy vertex ordering. The bounds obtained on that algorithm may be of independent interest, as well as the scaling properties derived. The hardness results utilize the result of Feige and Kilian [5] of the hardness of computing the chromatic number of a graph. That result is based on the complexity-theoretic assumption $NP \neq ZPP$, that NP does not have polynomial-

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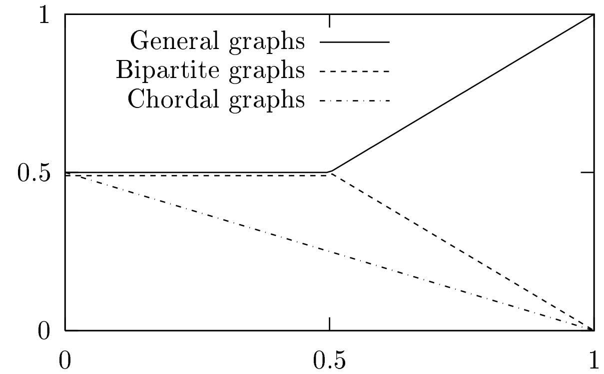

Figure 1: The optimal performance functions for (h, k)-coloring, with $\log_n h$ on x-axis and $\log_n \rho(h)$ on the y-axis

time randomized algorithms. We shall say that a computation problem is *hard* if there exists no polynomial-time randomized algorithm for the problem, unless $NP = ZPP$.

1.2 Previous results

Previous research on (h, k)-coloring problems includes both exact and inexact bounds on special classes of graphs, and hardness proofs; see the survey of Calamoneri [3]. The constructive upper bounds for coloring special classes of graphs can be viewed as approximation algorithms, while relative approximation results appear to be rare.

Few approximation results exist on general graphs, or on classes without constant factor upper bounds. Mostly, these are restricted to the $(1, 1)$-coloring problem, known as the *distance-2 coloring problem* (and closely related to coloring the square of a graph [2]). McCormick [8] showed that a greedy algorithm attains an $O(\sqrt{n})$-approximation (see also [1]). Agnarsson, Greenlaw, and Halldórsson [1] showed that the problem is hard to approximate within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$. This hardness holds also in the case of bipartite graphs and split graphs.

Few works have been given on approximation algorithm for (h, k)-coloring, for other values of h and k. Calamoneri and Vocca gave [4] an $h\sqrt{n}(1 + o(n))$-approximation algorithm for (h, k)-coloring with $h > k$, as well as approximations of bipartite graphs that are asymptotically $\min(h, 2k)\sqrt{n}$ and $(4/3)\Delta^2$ factors. They observe an equivalence in approximating the (h, k)-coloring problem and the $(1, 1)$-coloring problem, within a linear factor of h. That implies, for instance, $n^{1/2-\epsilon}$ approximation hardness for (h, k)-coloring bipartite graphs, for h constant, using a result of [1]. Our results extend this to general values of h.

The results of this work are given in the following section. We first derive some properties of rounding the separation requirements, and then analyze the First-Fit algorithm. We use these in the following subsections to give upper and lower bounds for approximating (h, k)-colorings in bipartite graphs, general graphs, and chordal and split graphs.

2 Approximation Results

The *span* of an (h, k)-coloring ψ is the value of the largest color used, plus one[1], i.e. $\max_{v \in V(G)} \psi(v) + 1$. Let $\lambda_{h,k}(G)$ denote the minimum span of an (h, k)-coloring of a graph G. The performance ratio ρ_A of an (h, k)-coloring algorithm A is the maximum ratio between the maximum and minimum spans, i.e.,

$$\rho_A = \rho_A(n) = \max_{G, |V(G)|=n} \frac{A(G)}{\lambda_{h,k}(G)}.$$

Recall that $\chi(G)$ denotes the minimum number of colors in an ordinary coloring of graph G, and $\alpha(G)$ is the size of the maximum independent set of G.

Let $d(v)$ be the number of neighbors of a vertex v and let $\Delta = \Delta(G)$ be the maximum degree of the graph. Let $d_2(v)$ be the number of vertices of distance 2 from a vertex v and $\Delta_2 = \max_v d_2(v) \leq \Delta(\Delta - 1)$ be the maximum of these values over the vertices in the graph. Let n be the number of vertices in the graph.

2.1 Scaling properties

We first simplify the problems by showing that it suffices to consider only a restricted subset of possible colorings, and that we can omit the factor k with only a small loss of performance.

It is well known that by uniformly increasing the gap between the vertices, one obtains a proper coloring with larger separations.

Observation 2.1 *Consider an (h, k)-coloring ψ with span λ. Then, the coloring ψ', given by $\psi'(v) = \psi(v) \cdot t$, is an $(h \cdot t, k \cdot t)$-coloring with span $(\lambda - 1) \cdot t + 1$.*

The converse holds also when the two separation constraints have a common divisor.

Lemma 2.2 *Consider an $(h \cdot t, k \cdot t)$-coloring ψ with span λ. Then, the coloring ψ', given by*

$$\psi'(v) = \left\lfloor \frac{\psi(v)}{t} \right\rfloor,$$

is a valid (h, k)-coloring with span $\lfloor (\lambda - 1)/t \rfloor + 1$.

Proof. Suppose there are vertices u, v whose colors $\psi'(u)$, $\psi'(v)$ falsify the claim. Then, either u and v are adjacent and $|\psi'(u) - \psi'(v)| \leq h - 1$, or u and v share a common neighbor and $|\psi'(u) - \psi'(v)| < k$. Consider the former case; the other is identical and will be omitted. Write $\psi(u) = t \cdot \psi'(u) + r_v$ and $\psi(v) = t \cdot \psi'(v) + r_u$, for some $0 \leq r_v, r_u < t$. Then, $|\psi(u) - \psi(v)| = |t \cdot (\psi'(u) - \psi'(v)) + (r_u - r_v)| \leq t \cdot |\psi'(u) - \psi'(v)| + |r_u - r_v| < t(h-1) + t = th$. Then, u and v are not properly (ht, kt)-colored. This is a

[1] Note that the span is frequently defined to be simply the largest color used. Our definition matches the size of the color palette used, including the "holes". The difference is not significant for the approximation results presented here.

contradiction; hence, the coloring ψ' is valid. The largest color under ψ is $\lambda - 1$, so the largest color under ψ' is $\lfloor (\lambda-1)/t \rfloor$. □

When there is no common divisor, one can create one by rounding up the values with a small increase in the span.

Lemma 2.3 *For any integers h, k, consider an (h,k)-coloring ψ with span λ. Then, for any integer t,*

$$\psi'(v) = \psi(v) + \lfloor \psi(v) \cdot t/h \rfloor$$

is a valid $(\lceil h/t \rceil t, k)$-coloring with span at most $(1+t/h)\lambda$.

Proof. Let u and v be adjacent vertices and suppose u receives the larger color of the two by ψ. So, $\psi(u) \geq \psi(v) + h$. Then the separation of the vertices under ψ' is at least

$$\begin{aligned} \psi'(u) - \psi'(v) &= \psi(u) - \psi(v) + \lfloor \psi(u)t/h \rfloor - \lfloor \psi(v)t/h \rfloor \\ &\geq h + \lfloor (\psi(v)+h)t/h \rfloor - \lfloor \psi(v)t/h \rfloor \\ &= h + t. \end{aligned}$$

Furthermore, $h+t \geq \lfloor (h+t)/t \rfloor t \geq \lceil h/t \rceil t$. Also, any pair of vertices is separated by no lesser amount under ψ' than under ψ. Thus, ψ' is a valid $(\lceil h/t \rceil t, k)$-coloring.

The span of ψ' is the value of the largest color used plus one, or

$$\begin{aligned} ((\lambda-1) + \lfloor (\lambda-1)t/h \rfloor) + 1 &\leq \lambda + \lfloor \lambda t/h \rfloor \\ &= \lfloor (1+t/h)\lambda \rfloor. \end{aligned}$$

□

Corollary 2.4 *The (h,k)-coloring problem is equally hard to approximate as the $(\lceil h/k \rceil, 1)$-coloring problem, within a factor of 2.*

Proof. Use the preceding two lemmas, with $t = k$ in the second lemma, we see that An (h,k)-coloring with span λ can be turned in polynomial time into a $(\lceil h/k \rceil, 1)$-coloring with span at most $2\lambda/k$. Also, by Lemma 2.2, a $(\lceil h/k \rceil)$-coloring with span λ' can be turned in polynomial time into a $(\lceil h/k \rceil k, 1)$-coloring, which also is an (h,k)-coloring, with span $\lambda' k$. Thus, any transformation between the two problems can only lose a factor of at most 2. □

In particular, (h,k)-coloring problems with $h < k$ reduce to the $(1,1)$-coloring problem within a constant approximation factor.

2.2 Analysis of First-Fit

The First-Fit (FF) algorithm is one of the simplest coloring strategies. Processing the vertices in an arbitrary order, each vertex is assigned the smallest color compatible with its neighborhood. For the (h,k)-coloring problem, that means satisfying the distance constraints to the previously colored neighbors as well as previously colored vertices of distance two.

First-Fit is an online algorithm, so the upper bounds proven also give upper bounds on the competitive ratio of online coloring algorithms. It can also be a component of a distributed strategy, when complemented by a synchronization primitive.

Lemma 2.5 *The span of a First-Fit (h,k)-coloring of a graph G is at most*

$$\begin{aligned} FF(G) &\leq \max_{v \in V} [(d_2(v) - d(v) \cdot (2k-1) + d(v) \cdot (2h-1)] \\ &\leq \Delta_2 \cdot (2k-1) + \Delta \cdot (2h-2k) + 1. \end{aligned}$$

Further, $FF(G) \leq (n-1) \cdot h + 1$.

Proof. Each neighbors u of v can cause at most $2h-1$ colors to be unavailable for v to use: $h-1$ above, $h-1$ colors below, and then the color u. Similarly, the $d_2(v) - d(v)$ distance-2 neighbors of v that are not neighbors of v can each make $2k-1$ unavailable. Finally, there is the single color used by v. □

Lemma 2.6 *For any graph G, the minimum span of an (h,k)-coloring of G is bounded below by*

$$\lambda_{h,k}(G) \geq (\Delta - 1) \cdot k + h + 1.$$

Proof. Each of the Δ neighbors of a maximum degree vertex v, as well as v itself, must be mutually k colors apart, using at least $\Delta k + 1$ colors. The separation from v to its nearest colored neighbor must be an additional $h - k$. □

Theorem 2.7 *The performance ratio of First-Fit, denoted as ρ_{FF}, is at most $O(\min(\Delta, h/k + \sqrt{n}))$. Furthermore, this is tight within a constant factor, for any combination of the parameters, even in the case of bipartite graphs.*

Proof. By Corollary 2.4, we may assume without loss of generality that $k = 1$. Let G be a graph with n vertices and maximum degree Δ, $FF(G)$ be the span of a First-Fit $(h,1)$-coloring of G, and $\lambda_{h,1}(G)$ be the minimum span. Let $\rho_{FF} = \max_G \frac{FF(G)}{\lambda_{h,1}(G)}$.

By Lemmas 2.5 and 2.6, we have that $FF(G) \leq \min(n, (\Delta-1)\Delta) + \Delta \cdot (2h-1) + 1$ and $\lambda_{h,1}(G) \geq \Delta + h$. Now, $(\Delta-1)\Delta/\Delta = \Delta - 1$ and $\Delta(2h-1)/h \leq 2\Delta$, so $FF(G)/\lambda_{h,1}(G) \leq 2\Delta$. Also, if $\Delta > h + \sqrt{n}$, we have that

$$\frac{FF(G)}{\lambda_{h,1}(G)} \leq \frac{n + \Delta(2h-1)}{\Delta} \leq \sqrt{n} + (2h-1).$$

To see that these bounds are tight, consider the bipartite graph $B_{m,m}$ which consists of a complete bipartite graph $K_{m,m}$ from which a perfect matching has been removed. Namely, $B_{m,m}$ contains vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m$, and edges (u_i, v_j) for all $i \neq j$. When the vertices are ordered $u_1, v_1, u_2, v_2, \ldots, u_m, v_m$, First-Fit will assign the colors $0, 0, h, h, \ldots, (m-1)h, (m-1)h$, while an optimal coloring uses colors $0, 1, \ldots, m -$

$1, m-1+h, m-1+h+1, \ldots, 2(m-1)+h$. The ratio between the two spans is at least $\min(h/2, m-1)$. By letting m range from $\sqrt{n}$ to $n/2$ and adding edges and degree-1 vertices to allow Δ to range from m to $n-m$, we obtain a tight bound for the second part of the claim. □

2.3 Bipartite graphs

The bound on First-Fit of Lemma 2.5 gives a good upper bound for bipartite graphs when $h/k \leq \sqrt{n}$. We now give a matching hardness result for this case.

Theorem 2.8 *The (h,k)-coloring problem with $h/k \leq \sqrt{n}$ is hard to approximate on bipartite graphs within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$.*

Proof. We use a hardness construction from [1] for the distance-2 coloring problem.

Given a graph G on N vertices, we construct a graph H that contains N copies $u_{i,\ell}$ of each vertex v_i in G along with additional vertices $x_1, x_2, \ldots, x_N$. Let $n = N^2 + N$ denote the number of vertices in H; thus, $N = \sqrt{n}(1 - o(1))$. A copy of vertex v_i is adjacent to x_j if and only if $\{v_i, v_j\}$ is an edge in G or if $i = j$. Formally, let

$$\begin{aligned} V(H) &= \{x_i, u_{i,j} : 1 \leq i,j \leq N\}, \text{ and} \\ E(H) &= \{\{x_i, u_{j,l}\} : \{v_i, v_j\} \in E(G) \text{ or } i = j\}\} \end{aligned}$$

Observe that vertices $u_{i,\ell}$ and $u_{j,k}$ must receive different colors in an $(h,1)$-coloring of H iff they are copies of the same or adjacent vertices in G. Thus, $\lambda_{h,1}(H) \geq n/\alpha(G)$. On the other hand, given a coloring ϕ of G, there exists a valid $(h,1)$-coloring of H formed as follows: Use the color $N \cdot \phi(v_i) + \ell$ on each vertex $u_{i,\ell}$, and the colors $(h + N \cdot \chi(G)) + \phi(v_j)$ on each vertex x_j. The $h + N \cdot \chi(G)$ term ensures a proper separation between adjacent vertices in H, the ℓ term ensures a separation between copies of the same vertex, and ϕ-terms ensures a separation between copies corresponding to distinct vertices in G. This coloring has a span of $h + (N+1)\chi(G)$; hence, $\lambda_{h,1}(G) \leq (N+1)\chi(G) + h$.

The hardness construction for graph coloring of Feige and Kilian [5] shows that for any $\epsilon > 0$, it is hard to distinguish between graph instances G on N vertices with the following two cases: a) $\alpha(G) \leq N^\epsilon$, and b) $\chi(G) \leq N^\epsilon$. When a) holds, then $\lambda_{h,1}(H) \geq n/N^\epsilon = \Omega(n^{1-\epsilon/2})$, while when b) holds, then $\lambda_{h,1}(H) \leq O(N^{1+\epsilon} + h) = O(n^{1/2+\epsilon/2} + h)$. Thus, it is hard to distinguish between graphs for which $(h,1)$-colorings require $O(n^{1/2+\epsilon/2} + h)$ colors or $\Omega(n^{1-\epsilon/2})$ colors. Thus, for any $h \leq \sqrt{n}$, we obtain a *gap* of $\min(n^{1/2-\epsilon}, n^{1-\epsilon}/h) = n^{1/2-\epsilon}$, for any $\delta > 0$. For $h \geq \sqrt{n}$, we obtain a gap of $\Omega(n^{1-\epsilon/2}/h)$. □

For larger values of h/k, the performance ratio decreases linearly.

Theorem 2.9 *For any h, k, possibly functions of n, the (h,k)-coloring problem can be approximated within a factor of $O(\min(\sqrt{n}, n/(h/k)))$ on bipartite graphs.*

Proof. By Corollary 2.4, we may assume without loss of generality that $k = 1$. Given a bipartite graph $G = (U, V, E)$, we color U and V separately, and separate the color sets by a distance of h. The coloring of each set corresponds to a distance-2 coloring of the induced subgraphs, which requires at most $\Delta_2 + 1$ colors. In total, the algorithm uses at most $2\Delta_2 + 2 + h \leq 2\Delta^2 + h$ colors, and trivially also at most n colors. Compared with the easy lower bound of $\Delta + h$, this gives a performance ratio of at most $\min(2\Delta, n/h, \sqrt{n})$. □

For $h \geq \sqrt{n}$, the reduction of the proof of Theorem 2.8 gives a lower bound of $\Omega(n^{1-\epsilon}/h)$.

2.4 General graphs

We shall assume in this section that $k = 1$, by reference to Corollary 2.4.

The results of Theorems 2.7 and 2.8 essentially conclude the cases where $h \leq \sqrt{n}$. For general graphs, we can obtain improved hardness results when h is large.

Theorem 2.10 *The $(h,1)$-coloring problem is hard to approximate within a factor of h/n^ϵ, for any $\epsilon > 0$ and $h \leq n$.*

Proof. Recall the Feige-Kilian gap. We show that on the same graph, we obtain a gap of nearly $\Omega(h)$.

When $\alpha(G) \leq n^\epsilon$, we have that any set of h consecutive colors of an $(h,1)$-coloring can contain at most $\alpha(G)$-vertices. Thus the span of the coloring is at least $nh/\alpha(G) \geq n^{1-\epsilon}h$.

When $\chi(G) \leq n^\epsilon$, we can construct an $(h,1)$-coloring by coloring the vertices in order of their ordinary color, using a new color for each vertex, but adding a separation of h when a new color class is considered. This gives a span of at most $n + (\chi(G) - 1)h \leq n + n^\epsilon h$. The gap between the two ratios is therefore at least $\min(n^{1-2\epsilon}, h/n^\epsilon)$. □

This hardness result is matched by the linear upper bound of Theorem 2.7 on the performance ratio of First-Fit for $h > \sqrt{n}$.

When h is huge, the $(h,1)$-coloring problem essentially reduces to ordinary graph coloring.

Lemma 2.11 *Consider the $(h,1)$-coloring problem for a graph G where $h \geq \Delta_2(G)$. Then, the $(h,1)$-coloring and the ordinary coloring problems are equivalent for G, within a constant factor.*

Proof. Consider an ordinary vertex coloring ψ of G that uses $\chi(G) \leq \Delta \leq h$ colors. Also, let ϕ be a proper distance-2 coloring of G, using at most $\Delta_2 + 1 \leq \min(\Delta(\Delta-1)+1, n)$ colors. Form an $(h,1)$-coloring ψ' of G by

$$\psi'(v) = 2h \cdot \psi(v) + \phi(v).$$

Then, vertices u and v assigned different color by ψ are separated by at least $2h - |\phi(u) - \phi(v)| \geq 2h - \Delta_2 \geq h$

colors under ψ'. Also, vertices of distance 2 have a different ϕ-value, and will therefore be assigned different colors in ψ'. The span of this coloring is at most $2h(\chi+1)$.

On the other hand, by Lemma 2.2, an $(h,1)$-coloring ψ' of G of span λ can be turned into an ordinary coloring by

$$\psi(v) = \lfloor 2\psi'(v)/h \rfloor$$

with span at most $2\lambda/h$. It follows that $\lambda_{h,1}(G) = \theta(h\chi(G))$. Hence, the approximabilities of graph coloring and $(h,1)$-coloring with $h \geq \Delta$ are equivalent within a constant factor. □

In particular, we can state that (h,k)-coloring is $O(n(\log\log n)^2/\log^3 n)$-approximable, given the best performance ratio currently known for graph coloring of [7].

2.5 Chordal graphs

A graph $G = (V,E)$ is *chordal* iff there is a *simplicial ordering* $v_1, v_2, \ldots, v_n$ of the vertices so that each vertex v_i forms a clique with its neigbors v_j of higher index, $j > i$. First-Fit is known to be an optimal coloring algorithm when applied on a reverse simplicial ordering of a chordal graph. We find that it also gives close to the best possible performance ratio for (h,k)-coloring chordal graphs.

Theorem 2.12 *First-Fit applied on a reverse simplicial ordering of a chordal graph attains a performance ratio of $O(\sqrt{nk/h})$ for (h,k)-coloring.*

Proof. In a reverse simplicial ordering, each vertex is preceded by at most $\omega-1$ of its neighbors, where $\omega = \chi(G)$ is the clique number of the chordal graph. Further, it is preceded by at most $\min(n, (\omega-1)(\Delta-1))$ distance-2 neighbors. As in previous arguments for FF, the color used on a vertex v is at most $2h-1$ times the number of previously colored neighbors plus $2k-1$ times the number of previously colored distance-2 neighbors plus one. Thus, the number of colors used is at most

$$FF(G) \leq 2h(\omega-1) + 2k\min(n, (\omega-1)(\Delta-1)) + 1.$$

An optimal $(h,1)$-coloring uses at least $h(\omega-1)+1$ colors (forced by a maximum clique), and also at least $k\Delta+1$ colors (forced by a largest neighborhood). This means that the performance ratio of FF is at most

$$\begin{aligned}\rho_{FF} &\leq \frac{2h(\omega-1)+1}{h(\omega-1)+1} + \frac{2k\min(n,(\omega-1)\Delta)}{\max(h(\omega-1), k\Delta)} \\ &\leq 2 + 2\min(\frac{n}{\Delta}, \frac{\Delta k}{h}) \\ &\leq 2 + 2\sqrt{nk/h},\end{aligned}$$

where the last inequality is obtained by taking the geometric mean of the two terms. □

We give a nearly matching lower bound, again modifying the hardness construction of [1]. It holds for a restricted subclass of chordal graphs called split graphs. A graph is a split graph if its vertex set is a union of a clique and an independent set.

Theorem 2.13 *The $(h,1)$-coloring problem is hard to approximate on split graphs within a factor of $(n/h)^{1/2-\epsilon}$, for any $\epsilon > 0$.*

Proof. Given a graph G on N vertices, construct a graph H that contains hN copies $u_{i,\ell}$ of each vertex v_i in G along with an additional clique on N vertices $x_1, x_2, \ldots, x_N$. Let $n = hN^2 + N$ denote the number of vertices in H; thus, $N = \sqrt{n/h}(1-o(1))$. A copy of vertex v_i is adjacent to the j-th clique vertex x_j if and only if $\{v_i, v_j\}$ is an edge in G or if $i = j$.

Observe that vertices $u_{i,\ell}$ and $u_{j,k}$ must receive different colors in an $(h,1)$-coloring of H iff they are copies of the same or adjacent vertices in G. Thus, $\lambda_{h,1}(H) \geq n/\alpha(G)$. On the other hand, given a coloring ϕ of G, there exists a valid $(h,1)$-coloring of H formed as follows: Use the color jh on vertex x_j, for $j = 1, \ldots, N$, and the color $Nh + Nh \cdot \phi(v_i) + \ell$ on vertex $u_{i,\ell}$, for each $i = 1, \ldots, N$ and $\ell = 1, \ldots, Nh$. This coloring has a span of $2Nh + Nh\chi(G)$; hence, $\lambda_{h,1}(G) \leq Nh(\chi(G)+2)$.

If $\alpha(G) \leq N^\epsilon$, then $\lambda_{h,1}(G) \geq n^{1-\epsilon/2}$, while if $\chi(G) \leq N^\epsilon$, then $\lambda_{h,1}(G) \leq h(n/h)^{1/2+\epsilon/2}$. Hence, by the Feige and Kilian result, there is an approximation gap of $(n/h)^{1/2-\epsilon}$. □

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