Sorting algorithms and permutation patterns

Computer and Information Sciences departmental seminar

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October 19, 2011

Table of Contents

Permutations

A permutation is a bijection $\pi\colon \{1,\dots,n\} \to \{1,\dots,n\}$ for some n.
We use one-line notation for permutations.

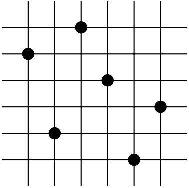
$$\pi = 526413$$

is the permutation that sends

$$\begin{aligned} 1 &\mapsto 5 \\ 2 &\mapsto 2 \\ 3 &\mapsto 6 \\ 4 &\mapsto 4 \\ 5 &\mapsto 1 \\ 6 &\mapsto 3 \end{aligned}$$

Drawing permutations

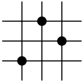
We can draw the graph of a permutation by placing dots on a grid.

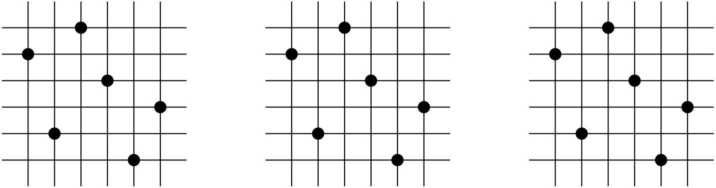
$526413 =$ 

Classical patterns

Patterns are permutations inside other permutations . . .

Example

The pattern  occurs in the permutation 526413.



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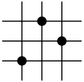
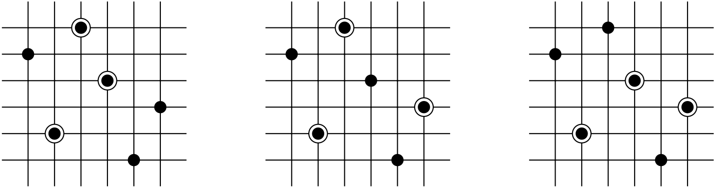
The pattern  occurs in the permutation 526413.



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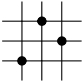
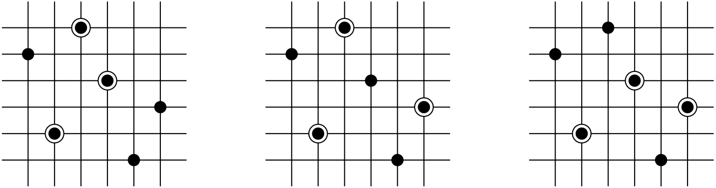
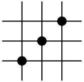
The pattern  occurs in the permutation 526413.



The same permutation avoids the pattern  .

Mesh patterns

Classical patterns form the base of a hierarchy of generalizations

mesh

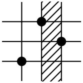
bivincular

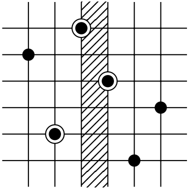
vincular

classical

Mesh patterns

Vincular patterns (2000) may require positions to be adjacent

The pattern  occurs in the permutation 526413



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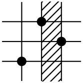
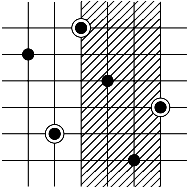
The pattern  occurs in the permutation 526413



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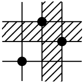
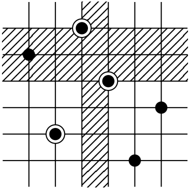
The pattern  is avoided by the permutation 526413



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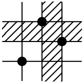
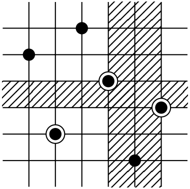
The pattern  is avoided by the permutation 526413



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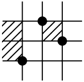
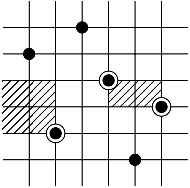
The pattern  occurs in the permutation 526413



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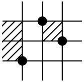
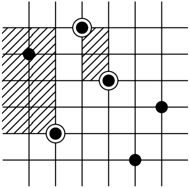
The pattern  occurs in the permutation 526413



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Simsun permutations

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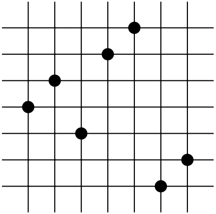

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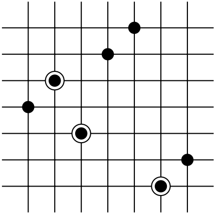

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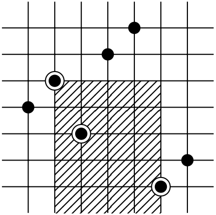

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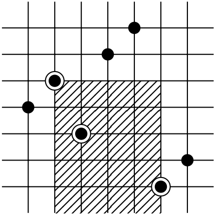
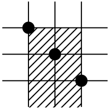
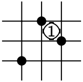


A permutation is simsun if and only if it avoids 

Marked mesh patterns

Marked mesh pattern (2010) restrict the number of elements in a region

The pattern 

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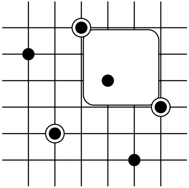

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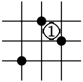
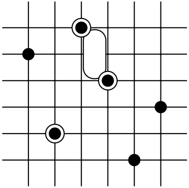
The pattern  occurs in the permutation 526413



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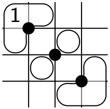
Freely braided permutations

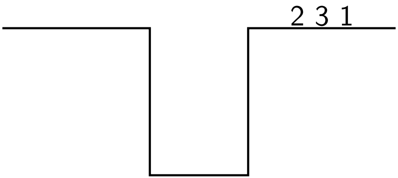
Green and Losonczy, 2002, defined freely braided permutations as those permutations avoiding the classical patterns 3421, 4231, 4312, 4321. Equivalently, these are the permutations avoiding



We now look at sorting and introduce a new type of pattern

Sorting with one stack

If we try to sort the permutation 231 with one stack . . .



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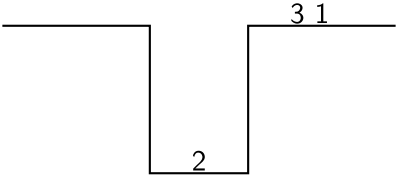

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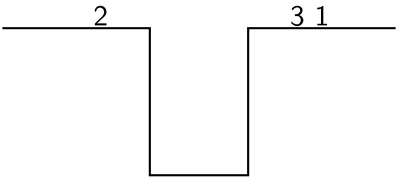

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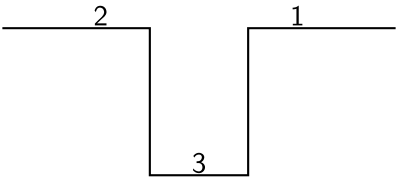

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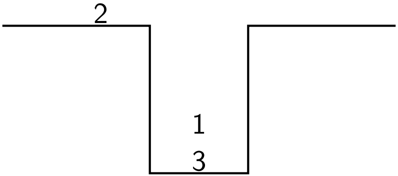

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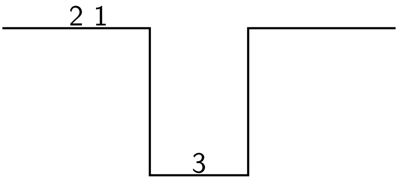

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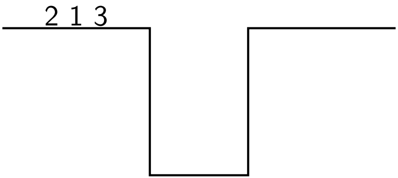

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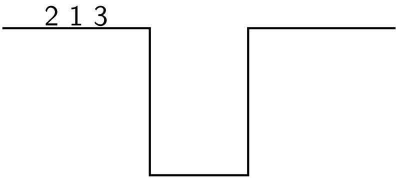

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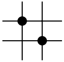
We failed because the outcome contained the pattern . The points making up that pattern must have come from a pattern

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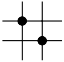
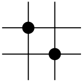
We failed because the outcome contained the pattern . The points making up that pattern must have come from a pattern



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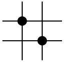
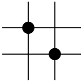
We failed because the outcome contained the pattern . The points making up that pattern must have come from a pattern



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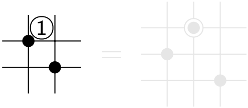

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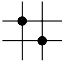
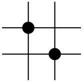
We failed because the outcome contained the pattern . The points making up that pattern must have come from a pattern



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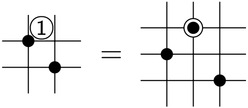

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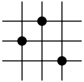
A permutation is not sorted in two passes if we pass it once through and the outcome contains the pattern . The points making up that pattern can come from either of the patterns

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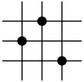
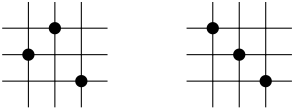
A permutation is not sorted in two passes if we pass it once through and the outcome contains the pattern . The points making up that pattern can come from either of the patterns



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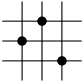
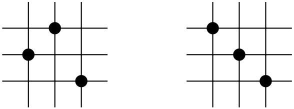
A permutation is not sorted in two passes if we pass it once through and the outcome contains the pattern  . The points making up that pattern can come from either of the patterns



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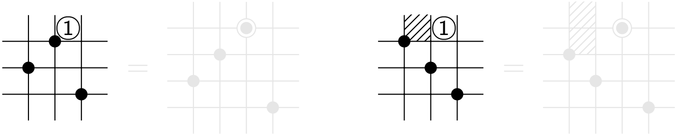

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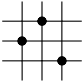
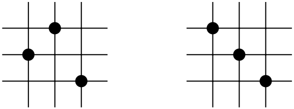
A permutation is not sorted in two passes if we pass it once through and the outcome contains the pattern . The points making up that pattern can come from either of the patterns



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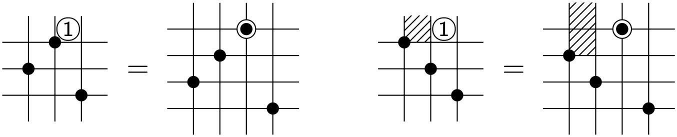

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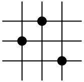
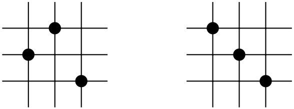
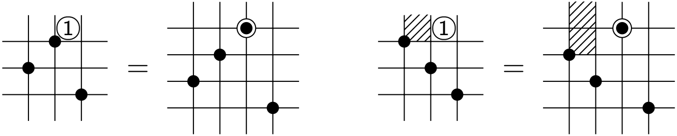
Characterizing permutations sortable in two passes

A permutation is not sorted in two passes if we pass it once through and the outcome contains the pattern . The points making up that pattern can come from either of the patterns



in the original permutation . . . and something pushed the big elements out of the stack before the small element could get on top and out of the stack



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Characterizing permutations sortable in two passes

Theorem (West 1990)

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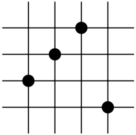

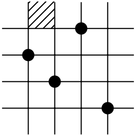

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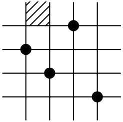

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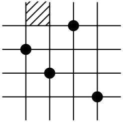

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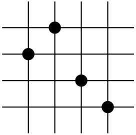

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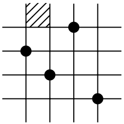

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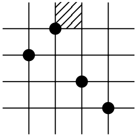

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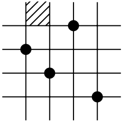

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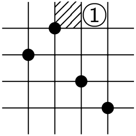

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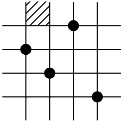

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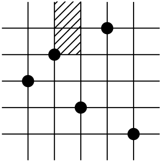

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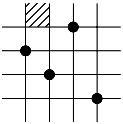

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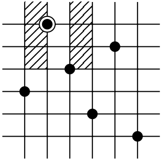

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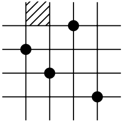

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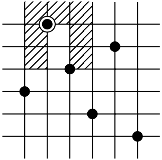

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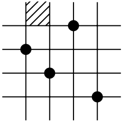

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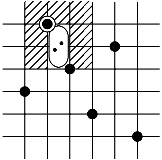

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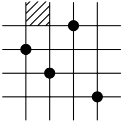

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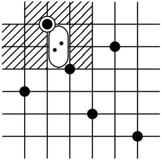
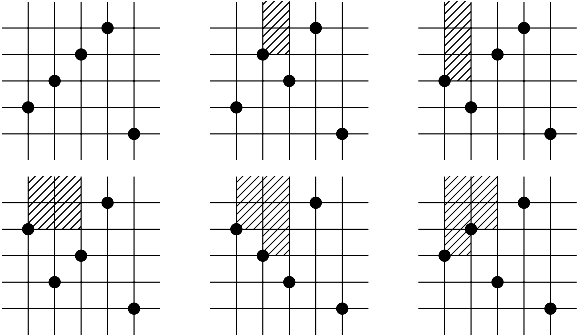


Characterizing permutations sortable in three passes

Theorem (Ú. 2011)

A permutation π is sortable in three passes if and only if it avoids the following decorated patterns



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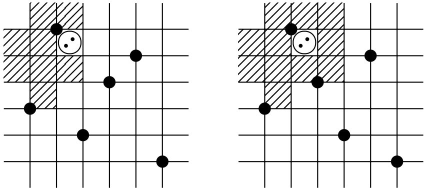

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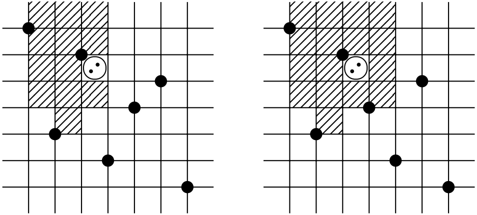

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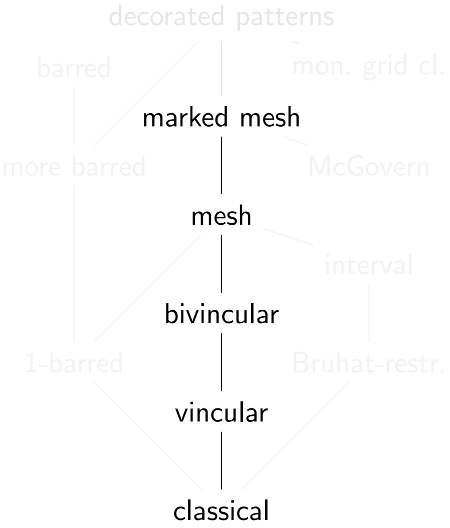
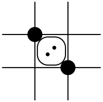
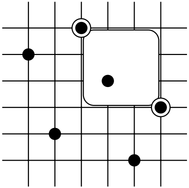


Decorated patterns

Decorated patterns (2011) allow even finer control over the elements in a region

The pattern  occurs in the permutation 526413.



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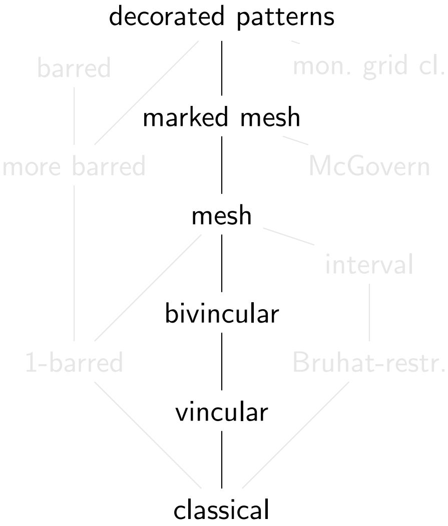

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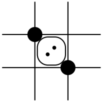
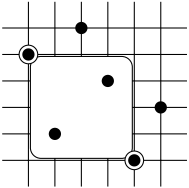
The pattern  occurs in the permutation 526413.



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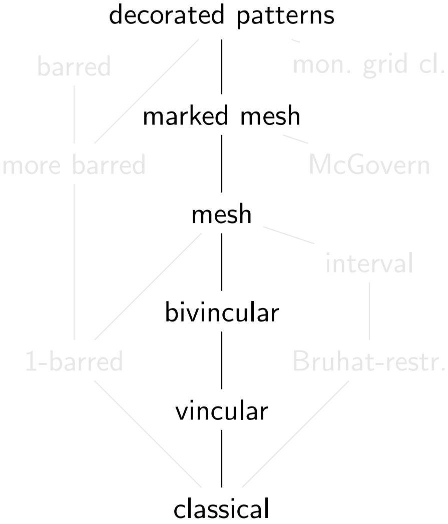

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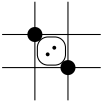
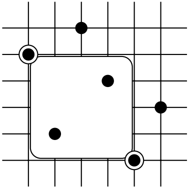
The pattern  occurs in the permutation 526413.



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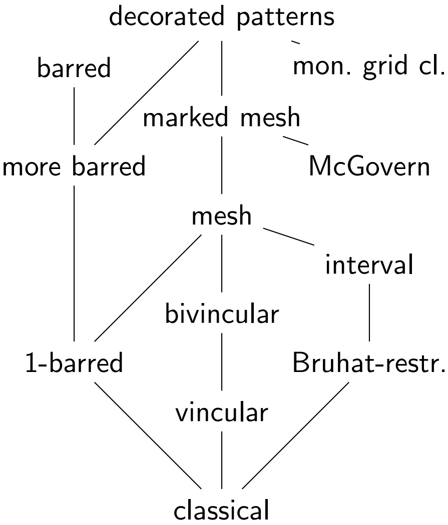
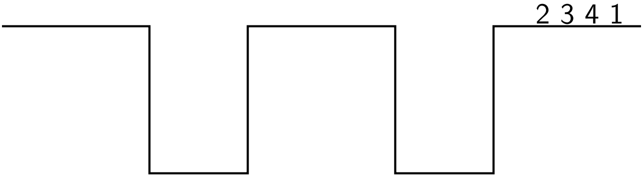


We end with some open problems.

Sorting with two stacks

We can sort the permutation 2341 with two stacks . . .



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We don't know how to characterize them by pattern avoidance.

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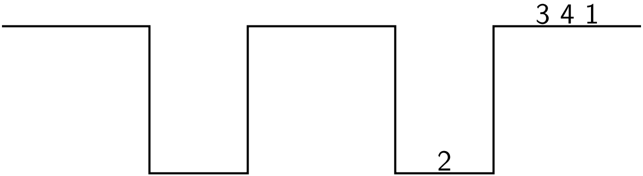

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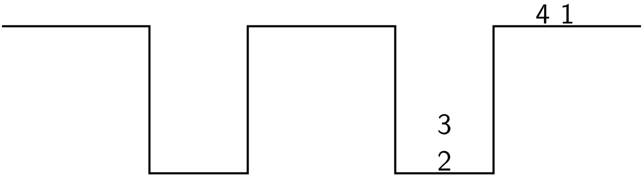

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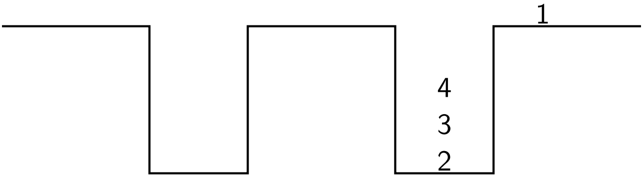

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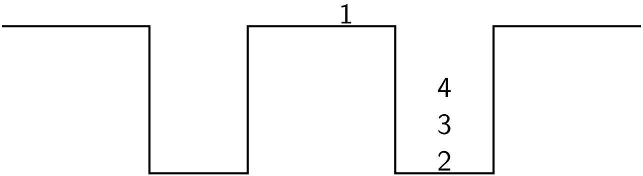

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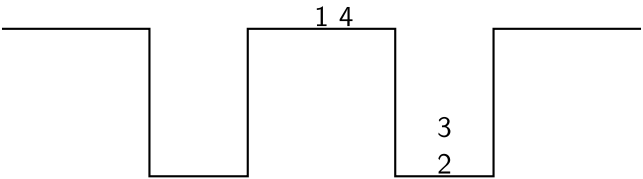

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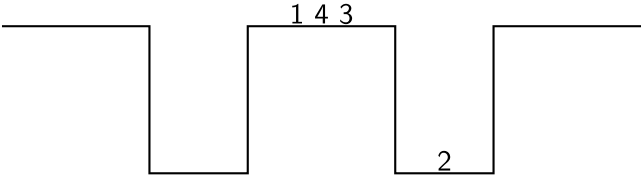

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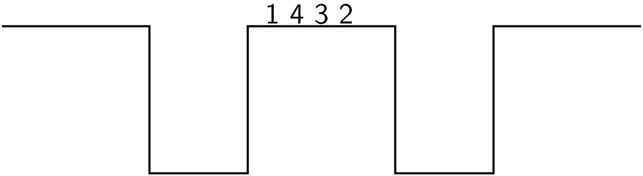

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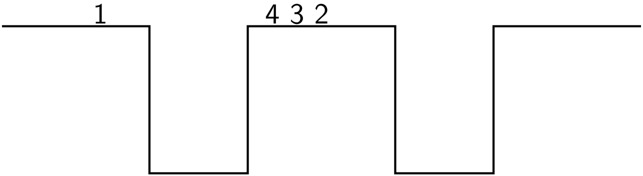

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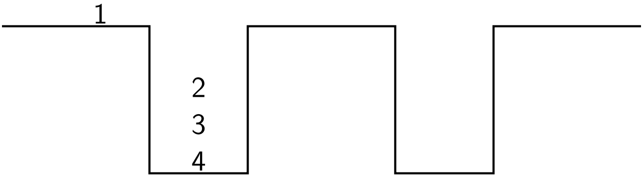

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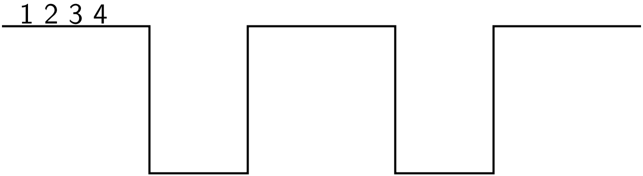

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Open Problem 1

Characterize permutations sortable with two stacks: wide-open and considered very hard.

Open Problem 2

Count permutations sortable with three passes through a single stack: wide-open and considered very hard. Maybe easier now because we have the patterns describing these permutations.

Open Problem 3

Can a computer do what we did? Can it figure out the patterns describing permutations sortable in four passes through a single stack?

Open Problem 4

Applications to other sorting operations. The method used above can easily be extended for the bubble-sort operator. How about others?

For more information

Describing West-3-stack-sortable permutations with permutation patterns
http://arxiv.org/abs/1110.1219

Thank you for listening!
Any questions?