RESEARCH STATEMENT

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Below I describe two research projects I am actively working on at the moment. The first is about equivalence relations on the set of permutations $\mathfrak{S}_n$ and pattern avoidance, and the second is about the interaction between algebraic geometry and algebraic combinatorics through Schubert varieties.

A section is devoted to each of the projects below. Each section is divided into an introduction and a more detailed description.

We start with a few basic definitions that we will need below.

Patterns in Permutations. We use one-line notation for all permutations, so we write $\pi = 132$ for the permutation that satisfies $\pi(1) = 1$, $\pi(2) = 3$ and $\pi(3) = 2$.

A *pattern* p is also a permutation, but we are interested in when and how a pattern is *contained* in a permutation π. An *occurrence* (or an *embedding*) of a pattern p in a permutation π is classically defined as a subsequence in π, of the same length as p, whose letters are in the same relative order (with respect to size) as those in p. For example, the pattern 1−2−3 corresponds to a increasing subsequence of three letters in a permutation, *that need not be adjacent.* If we use the notation 1_π for the first, 2_π for the second and 3_π for the third letter in an occurrence, then we are simply requiring that $1_\pi < 2_\pi < 3_\pi$. If a permutation has no occurrence of a pattern p we say that π *avoids* p.

Example 1. *The permutation* 32415 *contains two occurrences of the pattern* 1−2−3 *corresponding to the sub-words* 345 *and* 245. *It avoids the pattern* 1−3−2.

In a *vincular pattern*[1] two adjacent letters may or may not have a dash between them. The absence of a dash means that the corresponding letters in the permutation π must be adjacent.

Example 2. *The permutation* 32415 *contains one occurrence of the pattern* 12−3 *corresponding to the sub-word* 245. *It avoids the pattern* 1−23.

These types of patterns have been studied sporadically for a very long time but were not defined in full generality until Babson and Steingrímsson (2000).

This notion was generalized further in Bousquet-Mélou et al. (2008): In a *bivincular pattern* we are also allowed to put restrictions on the values that occur in an embedding of a pattern. The notation used for these patterns is (p, X, Y) where p is the pattern and the sets X and Y tell us which letters are supposed to be adjacent in position and value. This is best described by an example:

Example 3. *An occurrence of the pattern* $(123, \{1\}, \{2\})$ *in a permutation* π *is an increasing subsequence of three letters. Since we have* $1 \in X$ *we must have the first and the second letters adjacent in position and since we have* $2 \in Y$ *we must have the second and third letters adjacent in values. The permutation* 32415 *contains one occurrence of this bivincular pattern corresponding to the sub-word* 245. *The permutation avoids the bivincular pattern* $(123, \{1\}, \{1\})$.

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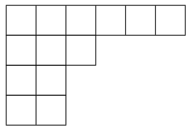
[1] Also called a *generalized pattern*, *Babson-Steingrímsson pattern* or a *dashed pattern.*

Young Tableaux. For a positive integer m we define a *partition* of it as a weakly decreasing sequence of positive integers $m_1, m_2, \ldots, m_r$ that sum to m. Note that 0 is not allowed in the sum. A partition can be depicted with a *Young diagram* (or *Ferrers diagram*) which has r rows of length $m_1, m_2, \ldots, m_r$, For example, the partition $13 = 6 + 3 + 2 + 2$ can be depicted with the diagram below.



The diagram is sometimes referred to as the *shape* of the partition. There is a very rich theory of Young diagrams where the boxes are labeled with integers according to specific rules. The diagrams are then referred to as *Young tableaux* and they have many applications in combinatorics, algebraic geometry and representation theory, Fulton (1997); Fulton and Harris (1991).

1. Equivalence Relations on Permutations and Patterns

Introduction. This project, which in its original form was suggested to me by Claesson and Steingrímsson, started as an investigation of the properties of permutations that avoid the pattern 2−3−1, especially which Young tableaux they correspond to under the RSK-correspondence. This correspondence gives a bijection between permutations and pairs of standard Young tableaux of the same shape. It is well known that permutations avoiding 1−2−3 correspond to pairs of tableaux with two or less columns and it was the objective to find a simple description for the other pattern as well.

The project changed course when I discovered that if we put an equivalence relation, called Knuth-equivalence, Knuth (1970), on the set of permutations, and look at entire equivalence classes that avoid the pattern 2−3−1, then a very simple description can be arrived at. This description allows one to count both the number of avoiding classes as well as the size of each class.

The project then grew considerably when I started trying other equivalence relations and I have now tried about 20 different equivalence relations from conjugacy to the location of recoils, and almost all of them give known sequences in The On-Line Encyclopedia of Integer Sequences, Sloane (2010). The way I have studied these equivalences is by running experiments in the computer algebra system Sage (2010) and searching the database of Sloane (2010). These methods have generated a database of empirical evidence for many conjectures which I have only just started working on.

Below I will describe two of these equivalence relations, *cycle type* and *toric equivalence* Eriksson et al. (2001), and some conjectures related to them, as well as some propositions that I have verified.

The second equivalence, toric equivalence, gives sequences that relate pattern avoidance to common functions in number theory, such as Euler's totient function and to the sum-of-divisors function. This has a fun consequence, as it allows us to state the Riemann Hypothesis in terms of pattern avoidance (Conjecture 19).

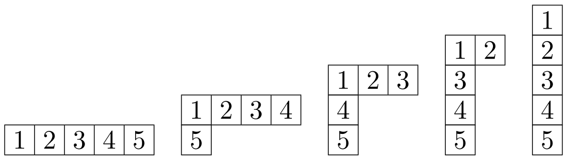
Details.

Knuth-equivalence. We will first describe the equivalence relation known as *Knuth-equivalence* which was first defined by Knuth (1970). We say that two permutations are *Knuth-equivalent* if and only if they can be connected by a sequence of *elementary transformations.* Here an elementary transformation corresponds to swapping neighbors if an adjacent entry is between them in values,

$$\cdots xyz \cdots \mapsto \cdots yxz \cdots, \quad \text{if } z \in [x, y],$$
$$\cdots zxy \cdots \mapsto \cdots zyx \cdots, \quad \text{if } z \in [x, y].$$

We write $\pi \equiv \pi'$ if the permutations π and π' are Knuth-equivalent. For example $41523 \equiv 14253$ because $41\mathbf{523} \equiv 41\mathbf{253}$ (since $2 < 3 < 5$) and then $\mathbf{41}253 \equiv \mathbf{14}253$ (since $1 < 2 < 4$). It then turns out that two permutations are Knuth-equivalent if and only if they have the same *insertion tableaux* (the first tableaux in the pair) under the RSK-correspondence.

We say that an equivalence class *avoids* a pattern p if every member of the class avoids p. The equivalence classes that avoid the pattern 2–3–1 correspond to very simple tableaux that are shaped like a hook and labeled by reading the rows from left to right. Here are the tableaux corresponding to the avoiding equivalence classes in $\mathfrak{S}_5$.



Since the avoiding classes correspond to such easily described tableaux it becomes easy to count how many classes there are: n classes in $\mathfrak{S}_n$, and the number of permutations in each class[2]: $\binom{n-1}{k}$ in the k-th class. If we put these two facts together we see that the union of the classes has exactly 2^{n-1} permutations in $\mathfrak{S}_n$.

It then turns out that this union of avoiding classes can also be described in terms of pattern avoidance without using an equivalence relation. The union is exactly $\mathfrak{S}_n(2\text{–}3\text{–}1, 2\text{–}1\text{–}3)$, the permutations in $\mathfrak{S}_n$ avoiding the patterns 2–3–1 and 2–1–3. One should notice here that these two patterns make up an equivalence class in $\mathfrak{S}_3$. The pattern 2–3–1 is therefore an example of a *stable* pattern, one that satisfies

$$\{\pi \in \mathfrak{S}_n \mid \pi, \text{ and every equivalent permutation, avoids the pattern } p\}$$
$$=$$
$$\{\pi \in \mathfrak{S}_n \mid \pi \text{ avoids the pattern } p \text{ and every equivalent pattern}\}$$

So by considering Knuth-equivalence we get a new proof of the fact that

$$\#\mathfrak{S}_n(2\text{–}3\text{–}1, 2\text{–}1\text{–}3) = 2^{n-1}.$$

This is well known (Simion and Schmidt (1985)), but this new proof can potentially be applied to other patterns whose avoiding classes have nice descriptions in terms of Young tableaux. Before we move on to other equivalence relations, let me point out that there are some *unstable* patterns, although you must go up to $\mathfrak{S}_4$ to find them. Here's one: 3–4–1–2.

Everything mentioned above is proven in the preprint Úlfarsson (2009a). I'm currently working on incorporating some of the material below into this preprint.

Other equivalence relations. It is now very natural to consider other equivalence relations on $\mathfrak{S}_n$. To date I have tried about 20 different equivalence relations and in most[3] cases there have appeared known sequences from Sloane (2010).

[2] Here one can use the hook-length formula of Frame et al. (1954), but in this simple case it is not really necessary. The formula might be needed for other patterns whose avoiding classes are not as simple.

[3] The only equivalence relation that has not given any interesting sequences so far is *entropy*. The *entropy* of a permutation is defined as the sum $\sum(\pi_i - i)^2$ and we say that two permutations

The way I have looked for sequences is by running the computer algebra system Sage (2010) to test avoidance up to about $\mathfrak{S}_{10}$ and then used Sloane (2010) to see if known sequences are produced. This has generated a lot of interesting data that I am only starting to go through and proving the conjectures that the empirical evidence suggest. Below I will describe two of these equivalence relations, *conjugacy* and *toric equivalence*. First we fix the following symbol,

$$\widetilde{\mathfrak{S}_n}(p) = \{\pi \in \mathfrak{S}_n \mid \pi, \text{ and every equivalent permutation, avoids the pattern } p\}.$$

Note that this is the union of the avoiding classes.

Conjugacy. With this relation we say that two permutations are equivalent if they are conjugate, that is, if they have the same cycle type. An example of two conjugate permutations from $\mathfrak{S}_4$ is $2143 = (12)(34)$ and $4321 = (14)(23)$ because both have the cycle type $(2, 2)$. A fun aspect of this equivalence relation is that it gives non-trivial counts even when we consider conjugacy classes avoiding a pattern of length 1:

Proposition 4.

$$\widetilde{\mathfrak{S}_n}(1, \{0\}, \{0\}) = \text{ derangements in } \mathfrak{S}_n.$$

Here a derangement *is a permutation with no fixed points. This gives the sequence A000166 in Sloane (2010).*

Next we consider three bivincular versions of the classical pattern 2−1:

Proposition 5.

$$\#\widetilde{\mathfrak{S}_n}(21, \{1\}, \{0, 2\}) = \binom{n}{2} + 1, \qquad n \geq 3.$$

The enumeration is explained by the fact that the set on the left contains only permutations with cycle type $(1, 1, \ldots, 1)$ *or* $(2, 1, \ldots, 1)$*. This gives the sequence A134869 in Sloane (2010).*

Proposition 6.

$$\begin{aligned} &\widetilde{\mathfrak{S}_n}(21, \{0, 1\}, \{0, 1\}) \\ &= \text{ permutations whose cycle decomposition has no transposition.} \end{aligned}$$

The enumeration is given by the expansion of $\frac{e^{-x^2/2}}{1-x}$*. This gives the sequence A000266 in Sloane (2010).*

Proposition 7.

$$\widetilde{\mathfrak{S}_n}(21, \{0, 1\}, \{0, 2\}) = \text{ involutions in } \mathfrak{S}_n.$$

Here an involution *is a permutation* π *with the property that* $\pi \circ \pi = 123 \cdots n$*, the identity. This gives the sequence A000085 in Sloane (2010).*

There are plenty of other interesting sequences that we get by considering patterns of length 3 and 4 but I have not proven any of them, so this is all I will say about conjugacy.

are *entropy-equivalent* if they have the same entropy. If we instead use the sum $\sum |\pi_i - i|$ (this is called the *sum-of-distances* of π) then we do get some interesting sequences.

Toric Equivalence. Before we get to toric equivalence we need a preliminary definition: If λ is a circular permutation of $[\![0, n]\!]$ then $\lambda_\circ$ denotes the permutation in $\mathfrak{S}_n$ we get by reading λ from 0.

Our definition of toric equivalence follows Eriksson et al. (2001), but an equivalent class of objects was studied by Steggall (1907). Here the equivalence can be roughly viewed as declaring two permutations to be equivalent if their permutation matrices become equal when they are wrapped around a torus. More precisely, given a permutation π of $\mathfrak{S}_n$ we define π° as the circular permutation 0π of $[\![0, n]\!]$. Then for any $m = 0, 1, \ldots, n$ we define a new circular permutation

$$\pi^\circ \oplus m = (0 + m)(\pi_1 + m)(\pi_2 + m) \cdots (\pi_n + m) \mod (n + 1).$$

(Here every letter is reduced modulo $n + 1$). Then the *toric class* of the original permutation π is defined as the set

$$\pi^\circ_\circ = \{(\pi^\circ \oplus m)_\circ \,|\, m = 0, 1, \ldots, n\}.$$

Example 8. *Let* $\pi = 1243$. *Then* $\pi^\circ = 01243$ *and*

$$\pi^\circ \oplus 0 = 01243, \quad \pi^\circ \oplus 1 = 12304, \quad \pi^\circ \oplus 2 = 23410,$$
$$\pi^\circ \oplus 3 = 34021, \quad \pi^\circ \oplus 4 = 40132.$$

The toric class is $\pi^\circ_\circ = \{1243, 4123, 2341, 2143, 1324\}$.

This equivalence relation behaves well with respect to the basic symmetries of the permutations $\mathfrak{S}_n$. For example it is not to hard too see that for any circular permutation λ of $[\![0, n]\!]$ we have

$$\lambda^{\mathrm{i}} \oplus x \equiv (\lambda \oplus 1)^{\mathrm{i}},$$

where x is the distance (counter-clock-wise) from n to 0 in λ.

Now, onto some pattern avoidances. We start by looking at the connections between toric equivalence and modular sequences in circular permutations.

Proposition 9.

1. *The set* $\widetilde{\mathfrak{S}_n}(1, \{0\}, \{0\})$ *is equinumerous with the set of circular permutations of* $[\![0, n]\!]$ *that have no modular* 2*-sequences. Here a* modular 2-sequence *is a substring of the form* $i(i + 1) \mod (n + 1)$. *This gives the sequence A000757 in Sloane (2010).*
2. *The set* $\widetilde{\mathfrak{S}_n}(12, \{0, 1\}, \{0, 1\})$ *is equinumerous with the set of circular permutations of* $[\![0, n]\!]$ *that have no modular* 3*-sequences. Here a* modular 3-sequence *is a substring of the form* $i(i + 1)(i + 2) \mod (n + 1)$. *This gives the sequence A165962 in Sloane (2010).*

It then turns out that you can keep going:

Proposition 10. *The set* $\widetilde{\mathfrak{S}_n}(12 \cdots n, \{0, 1, 2, \ldots, n - 1\}, \{0, 1, 2, \ldots, n - 1\})$ *is equinumerous with the set of circular permutations of* $[\![0, n]\!]$ *that have no modular* $(n + 1)$*-sequences. Here a* modular $(n + 1)$-sequence *is a substring of the form* $i(i + 1) \cdots (i + n) \mod (n + 1)$.

Now we turn to some conjectures about bivincular versions of the classical pattern 2−1−3 and the avoiding classes. The conjectures have been verified up to $\mathfrak{S}_9$, and while I have a strategy on how to prove them I haven't completed the proofs.

Conjecture 11. *For* $n \geq 2$

$$\#\widetilde{\mathfrak{S}_n}(213, \varnothing, \varnothing) = 2.$$

Conjecture 12. *For* $n \geq 1$

$$\#\widetilde{\mathfrak{S}_n}(213, \varnothing, \{1\}) = d(n),$$

where $d(n)$ *counts the number of divisors of* n. *This gives the sequence A000005 in Sloane (2010).*

Conjecture 13. *For* $n \geq 1$

$$\#\widetilde{\mathfrak{S}_n}(213, \varnothing, \{1, 3\}) = \phi(n+1),$$

where $\phi(n+1)$ *is the* Euler totient function*, which counts the numbers less than or equal to* $n+1$ *that are prime to* $n+1$. *This gives the sequence A000010 in Sloane (2010)*

I should note here that the bivincular patterns that appear in these three conjectures can be replaced by their inverses without changing the enumeration since toric equivalence interacts well with the three basic symmetries of $\mathfrak{S}_n$. Then we would instead have the patterns 2−1−3, 21−3 and 21−3| (an occurrence of the last pattern must end at the end of the permutation). The reason I do not use these easier to describe patterns is that it is easier to describe the permutations that avoid the original patterns.

In trying to prove the conjectures above one is lead to defining two classes of permutations (one inside the other).

Definition 14. Let $n \geq 1$ be fixed.

(1) For any k prime to $n+1$ we define a permutation $\nu_{k,n}$ first by constructing a circular permutation $\lambda_{k,n}$ of $[\![0, n]\!]$ as follows: Place 0 anywhere, then place 1 by moving k steps from 0 (so there are $k-1$ empty positions between 0 and 1), then place 2 by moving k steps from 1 and keep going until you place n. Then define $\nu_{k,n} = (\lambda_{k,n})_\circ$. We call the permutation constructed in this way a *natural permutation* (*corresponding to* k) in $\mathfrak{S}_n$.

(2) If k is a divisor of n we write $\delta_{k|n} = \nu_{k,n}$ and call $\delta_{k|n}$ a *divisor permutation* (*corresponding to* $k|n$) in $\mathfrak{S}_n$.

The condition that k be prime to $n+1$ is a necessary and sufficient condition for constructing $\nu_{k,n}$. The reason for calling these permutations *natural* is that the permutation $\nu_{k,n}$ behaves like the natural number k when multiplied with other natural permutations (see Property 3 below).

Example 15. *Let* $n = 6$. *Then the prime integers to* $n+1 = 7$ *are* $1, 2, 3, 4, 5$ *and* 6. *We construct* $\lambda_{1,6}$ *as follows:*

$$0\_\_\_\_\_\_ = 01\_\_\_\_\_ = 012\_\_\_\_ = \cdots = 0123456,$$

so $\nu_{1,6} = 123456 = \delta_{1|6}$. *Next we construct* $\lambda_{2,6}$*:*

$$0\_\_\_\_\_\_ = 0\_1\_\_\_\_ = 0\_1\_2\_\_ = 0\_1\_2\_3 = 041\_2\_3 = 04152\_3 = 0415263,$$

so $\nu_{2,6} = 415263 = \delta_{2|6}$. *Similarly we get*

$$\nu_{3,6} = 531642 = \delta_{3|6},$$
$$\nu_{4,6} = 246135,$$
$$\nu_{5,6} = 362514,$$
$$\nu_{6,6} = 123456 = \delta_{6|6}.$$

I now claim that these permutations are exactly the ones we are counting in Conjectures 12 and 13:

Conjecture 16 (Refinement of Conjecture 12)**.** *The set* $\widetilde{\mathfrak{S}_n}(213, \varnothing, \{1\})$ *consists of the divisor permutations in* $\mathfrak{S}_n$.

Conjecture 17 (Refinement of Conjecture 13)**.** *The set* $\widetilde{\mathfrak{S}_n}(213, \varnothing, \{1,3\})$ *consists of the natural permutations in* $\mathfrak{S}_n$*.*

The natural and divisor permutations seem to have many nice properties:

(1) By definition the location of 1 in $\nu_{k,n}$ and $\delta_{k|n}$ is k, or equivalently, the first letter of inverse of these permutations is k.
(2) They are the unique permutation in their toric class.
(3) They multiply like the natural numbers modulo $n+1$. For example

$$\delta_{2|6} \circ \delta_{2|6} = \nu_{4,6} \quad \text{and} \quad \nu_{4,6} \circ \nu_{5,6} = \delta_{6|6}.$$

(4) The the last letter in the divisor permutation $\delta_{k|n}$ is n/k, the *complementary divisor.*

Property (1) combined with Conjecture 16 gives us a way to write the *sum-of-divisors* function $\sigma(n)$ as

$$\sigma(n) = \sum_{\delta \in \widetilde{\mathfrak{S}_n}(213,\varnothing,\{1\})} (\text{location of 1 in } \delta).$$

Now consider the following theorem due to Robin (1984).

Theorem 18 (Robin's theorem)**.** *Let* $\sigma(n)$ *denote the sum of the divisors of* n*. The Riemann Hypothesis is true if and only if*

$$\sigma(n) < e^{\gamma} \log\log n,$$

holds for all n *larger than some constant. Here* γ *is Euler's constant.*

This allows us to state the Riemann Hypothesis in terms of pattern avoidance:

Conjecture 19 (Equivalent to RH modulo Conjecture 16)**.** *The Riemann Hypothesis is true if and only if*

$$\sum_{\delta \in \widetilde{\mathfrak{S}_n}(213,\varnothing,\{1\})} (\textit{location of } 1 \textit{ in } \delta) < e^{\gamma} \log\log n,$$

holds for all n *larger than some constant.*

The largest known n for which the inequality in Robin's theorem is violated is 5040, so it *suffices* to start exploring in $\mathfrak{S}_{5041}$. I should mention that permutations have been shown before to have connections with the Riemann Hypothesis, for example using the Redheffer matrix in Wilf (2004/06), probabilistic methods in Aldous and Diaconis (1999) (see also Stopple), and group theory in Massias et al. (1988).

Like I mentioned above I have a strategy for proving Conjectures (16), (17) and I'm optimistic that I will be able to produce proofs soon. I'm not as optimistic about proving the inequality in Conjecture (19).

Further questions and sub-projects.

(1) Given an equivalence relation, what patterns are stable? If we are working with Knuth-equivalence every classical pattern in $\mathfrak{S}_3$ is stable, but there are unstable patterns in $\mathfrak{S}_4$ and so far I have been unable to find a general characterization of the stable patterns. For other equivalence relations, like *circular equivalence* (permutations are equivalent if they are the same up to cyclic shifts) it is easy to see that every pattern (classical or bivincular) is stable.

(2) In all of these equivalence relations we almost always get non-interesting results (see Conjecture 11), when we restrict to classical patterns. We start getting more interesting sequences when we generalize to bivincular patterns. Kitaev and Remmel (2010) have a further generalization of bivincular patterns, called *place-difference-value patterns* (trivincular?). It would be very interesting to see what new sequences these patterns would produce.
(3) Given an equivalence relation, what can be said about the number of equivalence classes in $\mathfrak{S}_n$?
(4) What is the interplay between these relations and the basic symmetries. When the equivalence relation behaves nicely with respect to the symmetries it reduces the number of patterns one needs to look at. I have considered this question only for the toric equivalence and it turns out to respect all the basic symmetries as well as a new symmetry which produces non-trivial Wilf-equivalences for bivincular patterns.
(5) What other equivalence relations can one try? Like I have mentioned above I have tried about 20 relations and most of them hint at some interesting conjectures. An interesting option that I want to look at is to construct equivalence relations between permutations by using the properties of the Schubert varieties that they correspond to. See more about the relation between these varieties and permutations in the other project description.

2. Permutation Patterns and Schubert Varieties

Introduction. The goal of this project is to introduce bivincular patterns into the description of the smoothness properties of Schubert varieties. Schubert varieties are a very nice class of varieties and are often used as a test case for conjectures about more general varieties. These varieties are indexed by permutations and many properties of the varieties are encoded in the patterns that the permutations either contain or avoid. In particular the variety X_π, indexed by the permutation π, is non-singular if and only if π avoids $1{-}3{-}2{-}4$ and $2{-}1{-}4{-}3$ (Ryan (1987), Wolper (1989) and Lakshmibai and Sandhya (1990)). A weakening of non-singularity, called *factoriality* was then described in terms of so-called *barred patterns*. This project started when I noticed that this description would be greatly simplified by using bivincular patterns, which shows that weakening smoothness to factoriality corresponds to removing a dash, see Table 1. A further weakening called *Gorensteinness* can also be described in terms of bivincular patterns. This new description of factoriality and Gorensteinness in terms of bivincular patterns are described in detail in Úlfarsson (2009b).

Details. Not many Schubert varieties are non-singular and, in fact, the first result in this area, independently discovered by Ryan (1987), Wolper (1989) and Lakshmibai and Sandhya (1990) showed that the Schubert variety X_π is non-singular if and only if π avoids the patterns $1{-}3{-}2{-}4$ and $2{-}1{-}4{-}3$.[4] A weakening of non-singularity is the notion of a variety being *factorial*, this means that the local rings are unique factorization domains. Now, Bousquet-Mélou and Butler (2007) proved a conjecture by Yong and Woo (Bousquet-Mélou et al. (2005)) that factorial Schubert varieties are those that correspond to permutations avoiding $1{-}3{-}2{-}4$ and bar-avoiding $2{-}1{-}\overline{3}{-}5{-}4$. In the terminology of Woo and Yong (2006) the bar-avoidance of the latter pattern corresponds to avoiding $2{-}1{-}4{-}3$ with Bruhat

[4] It should be noted that the correspondence between the permutation π and the Schubert-variety X_π differs between authors, and in Lakshmibai and Sandhya (1990) the avoidance is in terms of the *complements* of the patterns we gave in the text. We have tried to convert everything here into a coherent notation.

condition ($1 \leftrightarrow 4$), or equivalently, interval avoiding $[2{-}4{-}1{-}3, 2{-}1{-}4{-}3]$ in the terminology of Woo and Yong (2008). But, as remarked in Steingrímsson (2007), bar-avoiding $2{-}1{-}\overline{3}{-}5{-}4$ is equivalent to avoiding the bivincular pattern $2{-}14{-}3$. If we summarize this in terms of bivincular patterns a striking thing becomes apparent; see Table 1.

Singularity property of X_π	The permutation π avoids the patterns
Non-singular	$1{-}3{-}2{-}4$ and $2{-}1{-}4{-}3$
Factorial	$1{-}3{-}2{-}4$ and $2{-}14{-}3$
Gorenstein	Description given below

Table 1. Connection between singularity properties and bivincular patterns.

By considering bivincular patterns we do not need to consider the more complicated barred pattern, Bruhat condition or the interval avoidance, because it becomes obvious that weakening non-singularity to factoriality corresponds to removing a dash from one of the patterns.

Woo and Yong (2006, 2008) also characterized Gorensteinness in terms of Bruhat conditions and interval avoidance and as I had hoped that Gorensteinness would correspond to removing more dashes from the classical patterns it turned out not to be the case. There are in fact two conditions for a Schubert variety being Gorenstein in terms of bivincular patterns:

Theorem 20 (Úlfarsson (2009b))**.** *Let $\pi \in S_n$. Then the Schubert variety X_π is Gorenstein if and only if*

(1) π is balanced; and
(2) the permutation π avoids the bivincular patterns

$$(31524, \{2\}, \{3\}) \text{ and } (24153, \{3\}, \{2\}).$$

This new description of factoriality and Gorensteinness in terms of patterns are described in detail in Úlfarsson (2009b).

I recently started a related project in this area with Woo, with the purpose of finding a description of *local complete intersection* varieties (whose definition I will omit) in terms of patterns. The project is still getting started but we have some conjectures that we are working on. Currently it seems that classical patterns might suffice to describe this property, but it will be a long list of patterns. It might then be possible to reduce that list to fewer bivincular patterns.

Further questions and sub-projects.

(1) While Gorensteinness can not be described by the removal of dashes from the original classical patterns $1{-}3{-}2{-}4$ and $2{-}1{-}4{-}3$ one can still ask what the removal of the individual dashes corresponds to on the geometry side. Could they give new singularity properties of Schubert varieties?
(2) Like I mentioned above I am starting to explore the local complete intersection property with Woo in terms of patterns. If we are successful then we would answer a question asked independently by Hasset, Joshua and Sturmfels.
(3) The Schubert varieties I have studied up to this point have all been algebraic subsets of the complete flag variety $\mathrm{Flags}(\mathbb{C}^n)$. There are results relating classical patterns to more general semi-simple simply-connected Lie groups via root systems, Billey and Postnikov (2005). When results on classical patterns are generalized to this setting they often become very complicated

or many patterns need to be used. It is likely that bivincular patterns play some role in this more general situation as well, and can hopefully be used to simplify the results as has happened with factoriality in the complete flag variety. A particular example is that a permutation in $\mathfrak{S}_n$ is *vexillary* if it avoids the pattern 2−1−4−3, Lascoux and Schützenberger (1985) but vexillary permutations in the hyperoctahedral group B_n are permutations that bar-avoid 18 different patterns, Billey and Lam (1998). It would be interesting to find a simpler description in terms of bivincular patterns.

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