RESEARCH STATEMENT

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My research interests lie in the expansive field of Algebraic Geometry. After I began working in Algebraic Geometry I have mostly worked on the following two projects:

Project 1: *The Serre Functor of Diagram Schemes* [UW09], [Ulf09a], [Ulf09b]

This project consists of three separate, but interconnected pieces:

Project 1a: ***Describing the Serre Functor*** [UW09]**.** [Joint work with J. Wise] This project started with my advisor's question about what the Serre functor looks like for *diagram schemes*, i.e., diagrams $X : S \to \mathsf{Sch}$ in the category of schemes[1]. Any smooth scheme Y has a dualizing sheaf ω_Y which gives us Serre duality in the bounded derived category of complexes of coherent sheaves on Y, $D^b(Y)$; i.e., for any two complexes E, F in $D^b(Y)$ we have a natural isomorphism

$$\mathbb{R}\mathrm{Hom}(E, F) \simeq \mathbb{R}\mathrm{Hom}(F, E \overset{\mathbb{L}}{\otimes} \omega_Y[n])^\vee,$$

where n is the dimension of Y. We define a functor $S : \mathsf{D^b}(\mathsf{Coh}Y) \to \mathsf{D^b}(\mathsf{Coh}Y)$, $S(E) = E \overset{\mathbb{L}}{\otimes} \omega_Y[n]$, and call it the *Serre functor*. The Serre functor is a powerfool tool for working with coherent sheaves and Lunts proved that a Serre functor exists for certain types of diagram schemes in [Lun01], but did not given a concrete description of it.

A simple example of a diagram scheme is the inclusion of a divisor into a scheme, $D \to Y$; this is given by the functor $X : \{\bullet_1 \to \bullet_2\} \to \mathsf{Sch}$, defined by $X(\bullet_1) = D$, $X(\bullet_2) = Y$ and $X(\bullet_1 \to \bullet_2)$ the inclusion of D into Y. I was able to give a description of the Serre functor for this type of diagram scheme with Jonathan Wise and we have a forthcoming preprint on the proof [UW09].

Future directions.

(1) Having given a description of the Serre functor for a simple diagram scheme like $D \to Y$ we want to extend our results to more general types of diagrams.
(2) When describing the Serre functor you end up working in the bounded derived category of complexes of coherent sheaves on the diagram, $D^b(D \to Y)$. We are hoping that inside this category there is a full subcategory that could be thought of as some kind of "relative derived category"; which should be the category of sheaves on Y whose cohomology sheaves are not supported on the divisor D in some sense. Finding such a subcategory might lead to interesting new derived equivalences[2].
(3) Lipman and Hashimoto have a forthcoming book titled "Foundations Grothendieck Duality for Diagram Schemes", [LH08], which might have some overlap with our work; I look forward to exploring the connection between the two approaches.

[1] Here X is a functor and S is any category, sometimes called the *type* of the diagram.

[2] Two schemes Y and Z are said to be *derived equivalent* if their bounded derived categories of complexes of coherent sheaves are equivalent, $D^b(\mathsf{Coh}Y) \simeq D^b(\mathsf{Coh}Z)$.

Project 1b: *The Underlying Category Theory* [Ulf09a]. Related to the description of the Serre functor for diagram schemes is the investigation of how the category of diagram schemes inherits properties from the category of schemes. In [Lun01] where Lunts proved the existence of the Serre functor he didn't construct a category around the diagram schemes, so I defined morphisms between diagram schemes and constructed the category $\mathsf{DiagSch}$[3]. Lunts defined and used *poset categories*[4] to describe sheaves on diagram schemes and I wanted to give a more natural description, in terms of fibered categories. When I had taken care of the above I wanted to ask if you can carry Grothendieck topologies and stacks[5] with you when you "diagrammatize", i.e., if you start with a site $(\mathsf{C}, \mathcal{T})$ and you form the diagram category DiagC, is there an extension of $\mathcal{T}$, to DiagC, call it $\mathsf{Diag}\mathcal{T}$, such that

(1) $\mathsf{Diag}\mathcal{T}$ restricted back to C is equivalent to $\mathcal{T}$.
(2) If $\mathcal{T}$ is subcanonical[6] then so is $\mathsf{Diag}\mathcal{T}$.
(3) If $\mathsf{F} \to \mathsf{C}$ is a stack in $\mathcal{T}$, then $\mathsf{DiagF} \to \mathsf{DiagC}$ is a stack in $\mathsf{Diag}\mathcal{T}$.

I found two candidate topologies for $\mathsf{Diag}\mathcal{T}$, one which satisfies only (1), and another which satisfies (1), (2) and (3). The latter one is constructed in such a way that "both the objects and the arrows in the diagrams are covered", without going into too much detail. I haven't been able to prove that this choice for $\mathsf{Diag}\mathcal{T}$ is the universal topology satisfying conditions (1), (2) and (3) above; but I'm working on it.

Future directions.

(1) I'm trying to show that the latter candidate for $\mathsf{Diag}\mathcal{T}$ mentioned above is the universal topology satisfying conditions (1), (2) and (3) above; or at least close to being it.
(2) Property (3) above can be philosophically thought of as follows: If X is a diagram in DiagC then the fiber over X in DiagF is obtained by gluing together the fibers in F that lie over $X(\alpha)$, as α runs over the set[7] of objects in S. I would like to extend this result to a result about gluing together topoi[8], in the spirit of [Ill71, Ill72].

Project 1c: *Homological algebra of shapes* [Ulf09b]. An outgrowth of Project 1b is a short note I wrote [Ulf09b], on how to extend the homological algebra of complexes to more general shapes. Let A be an abelian category and view the usual complexes as diagrams $A : \mathbb{Z} \to \mathsf{A}$, where we think about the integers $\mathbb{Z}$ as a category with an arrow $n \to m$ for every $n \leq m$. The main ingredient is still missing: the condition that two adjacent morphisms in the complex always compose to zero. Then our goal here is to try to impose this condition in a way that easily generalizes when we replace the integers $\mathbb{Z}$ with an arbitrary category S.

Here is how we go about doing this: There is a very natural *shift* functor, i.e., an auto-equivalence $[1]$ on $\mathbb{Z}$ that shifts everything to the right and requiring that every two adjacent morphisms in a complex $A : \mathbb{Z} \to \mathsf{A}$ compose to zero is equivalent to requiring that the composition $A(a[1]) \circ A(a)$ is the zero morphism for every arrow $a : n \to m$ that can be composed with its shift $a[1]$.

Now we can easily generalize this: Let S be a category with a collection of shift functors $\{\tau_i\}$. Then $A : S \to \mathsf{A}$ is called a *configured complex*, or *confplex*, if for any shift functor τ_i and any arrow a that can be composed with $\tau_i a$ we have

$$A(\tau_i a) \circ A(a) = 0.$$

[3] It turned out that this category had been used in [Ill71, Ill72], as well as many other places.
[4] A *poset category* is a collection of categories "glued glued together along a poset".
[5] A *stack* is a category fibered in groupoids and satisfying descent in some Grothenedieck topology
[6] A topology is *subcanonical* if every representable functor is a sheaf.
[7] Here we are implicitly assuming we are working with *small* categories; i.e., categories whose class of objects and arrows are sets.
[8] A *topos* is a category of sheaves on a Grothendieck site.

This gives us the objects of a category we write as $C_{\Diamond}(\mathsf{A})$. I will define the morphisms in this category here.

With this in our hands we can generalize familiar notions from homological algebra, such as cohomology groups, chain homotopies.

Future directions.

(1) I'm still working on how to extend the notion of a quasi-isomorphism to this setting, which will lead to a definition of a derived category whose objects are equivalence classes of generalized complexes.
(2) I briefly discussed this approach to homological algebra with Weibel, who mentioned that this might have connections to homological perturbation theory. I hope to establish and pursue that connection in the future.
(3) A similar project is underway with Joyce's series [Joy06, Joy07a, Joy07b, Joy08], and I would like to see whether my ideas have any relation to his.

Project 2: *Very Twisted Stable Curves in Gromov-Witten Theory* [CMU09]

[Joint work with Q. Chen and S. Marcus]

This project was suggested to my advisor by Jim Bryan and Rahul Pandharipande; earlier Ezra Getzler had suggested that very twisted curves should be of independent interest. I would like to thank Angelo Vistoli for his great help in understanding important pieces of the project.

Throughout, we let X be a smooth, projective Deligne-Mumford stack (DM-stack from now on) over an algebraically closed field k of characteristic 0.

The Gromov-Witten theory of orbifolds was first introduced in the symplectic setting in [CR05]. This was followed by an adaptation to the algebraic setting in [AGV02] and [AGV08], where the Gromov-Witten theory of DM-stacks was developed, and heavy use is made of the moduli stack of twisted stable maps into X, denoted $\mathcal{K}_{g,n}(X,\beta)$. This stack was constructed in [AV02] and is the necessary analogue of Kontsevich's moduli stack of stable maps for smooth projective varieties when replacing the variety with a DM-stack. The main purpose of this project is to provide a further extension of these spaces by allowing generic stabilizers on the source curves of the twisted stable maps.

Following [AGV08], we have a diagram:

$$\begin{array}{ccc} \Sigma_i^{\mathcal{C}} \subset \mathcal{C} & \xrightarrow{\ f\ } & X \\ \downarrow & & \\ \mathcal{K}_{g,n}(X,\beta) & & \end{array}$$

where $\Sigma_i^{\mathcal{C}} \subset \mathcal{C} \xrightarrow{f} X$ is the universal n-pointed twisted stable map. This gives rise to evaluation maps $e_i : \mathcal{K}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{I}}_\mu(X)$ mapping into the rigidified cyclotomic inertia stack $\overline{\mathcal{I}}_\mu(X)$ of X. If $\gamma_1,\dots,\gamma_n \in A^*(\overline{\mathcal{I}}_\mu(X))_{\mathbb{Q}}$ then the Gromov-Witten numbers are defined to be

$$\langle \gamma_1,\dots,\gamma_n \rangle_{g,\beta}^X = \int_{[\mathcal{K}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_i e_i^* \gamma_i$$

where $[\mathcal{K}_{g,n}(X,\beta)]^{\mathrm{vir}}$ is the virtual fundamental class of $\mathcal{K}_{g,n}(X,\beta)$ as in [BF97].

In the case when X is a 3-dimensional Calabi-Yau *variety*, we also have Donaldson-Thomas theory (originating from [Tho00],[DT98]) which in contrast to Gromov-Witten theory gives invariants by counting sheaves. We get the following diagram:

$$\begin{array}{c} \mathcal{Y} \subset X \times \mathcal{H}\mathrm{ilb}_{\chi,\beta}(X) \\ \downarrow \\ \mathcal{H}\mathrm{ilb}_{\chi,\beta}(X) \end{array}$$

where the virtual dimension of $\mathcal{H}\mathrm{ilb}_{\chi,\beta}(X)$ is zero as in [BF97]. The conjectural Donaldson-Thomas / Gromov-Witten correspondence of [MNOP06] predicts a correspondence (in the case n=0):

$$\left\{ \int_{[\mathcal{K}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1 \quad = GW(g,\beta) \right\} \longleftrightarrow \left\{ \int_{[\mathcal{H}\mathrm{ilb}_{\chi,\beta}(X)]^{\mathrm{vir}}} 1 \quad = DT(\chi,\beta) \right\}.$$

This correspondence is manifested in a subtle relationship between generating functions of the invariants. One wishes to discover a similar correspondence when X is a 3-dimensional Calabi-Yau *orbifold*. The Hilbert scheme of a stack was constructed by Olsson and Starr [OS03], but notice that in general it contains components corresponding to substacks with nontrivial generic stabilizers. In our definition of $\mathcal{K}_{g,n}(X,\beta)$ the twisted curves used as the sources of our maps have stacky structure only at the nodes and marked points. To allow for the above correspondence, one needs to extend the notion of the space of twisted stable maps as defined in [AV02] and [AGV08] so that our twisted curves have generic stabilizers. We approach this problem in three steps:

(1) Construct the stack $\mathcal{G}_X$ of étale gerbes in X as a rigidification of the stack $\mathcal{S}_X$ of subgroups of the inertia stack $\mathcal{I}(X)$. We exhibit $\mathcal{S}_X$ as the universal gerbe sitting over $\mathcal{G}_X$, giving a diagram:

$$\begin{array}{ccc} \mathcal{S}_X & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle \alpha} & & \\ \mathcal{G}_X . & & \end{array}$$

(2) Define the moduli stack of very twisted stable maps $\widetilde{\mathcal{K}}_{g,n}(X,\beta)$ by setting

$$\widetilde{\mathcal{K}}_{g,n}(X,\beta) := \coprod_{\beta_{\mathcal{G}}} \mathcal{K}_{g,n}(\mathcal{G}_X,\beta_{\mathcal{G}})$$

where the disjoint union is taken over all curve classes $\beta_{\mathcal{G}} \in H^*(\mathcal{G}_X)$ such that $\phi_*\alpha^*\beta_{\mathcal{G}} = \beta$. By pulling back, we see that each $\mathcal{K}_{g,n}(\mathcal{G}_X,\beta_{\mathcal{G}})$ has two different universal objects sitting above it, one giving twisted stable maps into $\mathcal{G}_X$ and the other giving "very twisted" stable maps into X. Through our disjoint union, we get the two corresponding universal objects sitting above $\widetilde{\mathcal{K}}_{g,n}(X,\beta)$.

(3) Relate the Gromov-Witten theory of X given by the two different universal objects sitting above $\widetilde{\mathcal{K}}_{g,n}(X,\beta) = \coprod_{\beta_{\mathcal{G}}} \mathcal{K}_{g,n}(\mathcal{G}_X,\beta_{\mathcal{G}})$ and show that they give the same invariants.

Although the naive GW-theory described above is the natural first approach to take, there is evidence from the DT-side of the conjectural correspondence for orbifolds that some adjustment to the theory is needed. In particular, the moduli spaces for orbifold Calabi-Yau 3-folds in the DT-theory have virtual dimension 0, and the correspondence would hope for the same to be true on the GW-side. An example sugested by Jim Bryan is the following. Let E be the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$ and consider the quotient $X = [E/(\mathbb{Z}/2\mathbb{Z})]$ where the action along each fiber is component-wise. X is an orbi-Calabi-Yau 3-fold with the zero-section giving an embedded $\widetilde{\mathbb{P}^1} = [\mathbb{P}^1/(\mathbb{Z}/2\mathbb{Z})]$. In this case, $\mathcal{S}_X = X \sqcup \widetilde{\mathbb{P}^1}$ and $\mathcal{G}_X = X \sqcup \mathbb{P}^1$. Stable maps into $\mathcal{G}_X$ (in particular, into $\mathbb{P}^1$) will not have virtual dimension zero. So even in this straightforward case we have a virtual fundamental class $\left[\widetilde{\mathcal{K}}_{g,n}(X,\beta)\right]^{vir}$ with non-zero virtual dimension. A nicer theory might adjust the relative obstruction theory used to construct the virtual fundamental class in order to fix the problem with the virtual dimension.

Future directions.

(1) Seeing whether the problem mentioned in the previous paragraph about the virtual dimension being non-zero can be resolved is very important; and is something we hope to look into.
(2) A very ambitious goal is to establish the GW-DT correspondence when X is a 3-dimensional Calabi-Yau orbifold, but one can at least hope to prove it in special cases.

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